



e-ISSN: 2319-8753 | p-ISSN: 2347-6710

# IJRSET

International Journal of Innovative Research in  
**SCIENCE | ENGINEERING | TECHNOLOGY**

# INTERNATIONAL JOURNAL OF INNOVATIVE RESEARCH

IN SCIENCE | ENGINEERING | TECHNOLOGY

Volume 10, Special Issue 2, December 2021

**ISSN** INTERNATIONAL  
STANDARD  
SERIAL  
NUMBER  
INDIA

**Impact Factor: 7.569**

9940 572 462

6381 907 438

[ijirset@gmail.com](mailto:ijirset@gmail.com)

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15<sup>th</sup> & 16<sup>th</sup> September 2021

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# Bounded Linear maps on Banach Spaces

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**ABSTRACT:** The purpose of this work to study a common policy for defining a bounded linear maps on Banach spaces are established, and their characterizations and properties are explored.

**KEYWORDS:** Quotient Space, totally bounded, bounded Linear Spaces, Extreme points, Banach spaces.

## I. INTRODUCTION

In functional analysis, a bounded linear operator is a linear transformation  $L : X \rightarrow Y$  between topological spaces  $X$  and  $Y$  that maps bounded subsets of  $X$  to bounded subsets of  $Y$ . If  $X$  and  $Y$  are normed vector spaces, then  $L$  is a bounded if and only if there exists some  $M \geq 0$  such that for all  $x$  in  $X$ ,  $\|Lx\| \leq M \|x\|$ . The smallest such  $M$ , denoted by  $\|L\|$ , is called the operator norm of  $L$ . [1]

### (1.1) Equivalence of boundedness and continuity [1].

As stated in the introduction, a linear operator  $L$  between normed spaces  $X$  and  $Y$  is bounded if and only if it is a continuous linear operator.

Proof:

Suppose that  $L$  is bounded. Then for all vectors  $x, h \in X$  with  $h$  non zero we have

$$\|L(x+h) - L(x)\| = \|L(h)\| \leq M \|h\|$$

Letting  $h$  goes to Zero shows that  $L$  is continuous at  $x$ . Since the constant  $M$  does not depend on  $x$ , implies that  $L$  is **uniformly continuous**, and even **Lipschitz continuous**. Conversely, it follows from the continuity at the zero vector that there exists a  $\delta > 0$  such that  $\|Lx\| = \|L(h) - L(0)\| \leq 1$  for all vectors  $h \in X$  with  $\|h\| \leq \delta$ , thus for all non zero  $x \in X$   $\|Lx\| = \left\| \frac{\|x\|}{\delta} L\left(\delta \frac{x}{\|x\|}\right) \right\| = \frac{\|x\|}{\delta} \left\| L\left(\delta \frac{x}{\|x\|}\right) \right\|$

### (1.2) Lipschitz continuous :

The condition for  $L$  to be bounded, namely that there exists some  $M$  such that for all  $x$ ,  $\|L(x)\| \leq M \|x\|$  is precisely the condition for  $L$  to be Lipschitz continuous at 0. Since  $L$  is Linear. [2]

## II. RELATED WORK

### PRELIMINARIES:

**Model 1:** Any linear operator between two finite-dimensional normed spaces is bounded, and such an operator may be viewed as multiplication by some fixed matrix.

**Model 2:** Any linear operator defined on a finite-dimensional normed space is bounded.

**Model 3:** On the sequence space  $C_{00}$  of eventually zero sequence of real number, considered with the  $\ell^1$  norm, the linear operator to the real number which returns the sum of a sequence is bounded, with operator norm 1. If the same space is considered with the  $\ell^\infty$  norm, the same operator is not bounded.

**Model 4:** Many integral transforms are bounded linear operator. For instance if

$K : [a,b] \times [c,d] \rightarrow \mathbb{R}$  is a continuous function, then the operator  $L$  defined on the space  $C[a,b]$  of continuous functions on  $[a,b]$  endowed with the uniform norm and with values in the space  $C[a,b]$  with  $L$  given by the formula  $(Lf)(y) = \int_a^b K(x,y)f(x)dx$  is bounded. This operator is in fact compact. The compact operator form an important class of bounded operators.



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**Model 15.** The Laplace operator  $\Delta: H^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$  (its domain is a Sobolev space and it takes values in a space of square-integrable function) is bounded.

**Model 16.** The shift operator on the  $\ell^2$  space of all sequences  $(x_0, x_1, x_2, \dots)$  of real numbers with  $x_0^2 + x_1^2 + x_2^2 + \dots < \infty$ .

$L((x_0, x_1, x_2, \dots)) = (0, x_0, x_1, x_2, \dots)$  is bounded. Its operator norm is easily seen to be 1.

### Unbounded linear operator [2]

Not every linear operator between normed space is bounded. Let  $X$  be the space of all trigonometric polynomials defined on  $[-\pi, \pi]$ , with the norm

$$\|P\| = \int_{-\pi}^{\pi} |P(x)| \, dx$$

Defined the operator  $L: X \rightarrow X$  which acts by taking the derivative, so it maps a polynomial  $P$  to its derivative  $P'$  then for  $v = e^{inx}$  with  $n = 1, 2, \dots$ . We have  $\|v\| = 2\pi n$  While  $\|L(v)\| = 2\pi n \rightarrow \infty$  as  $n \rightarrow \infty$ . so this operator is not bounded.

## III. METHODOLOGY

### FUNDAMENTAL PROPERTIES

#### Theorem 3.1:

Let  $X$  be a Metric Space. The Following conditions are equivalent

- (i)  $X$  is compact
- (ii) Every sequence in  $X$  has a convergent subsequence
- (iii)  $X$  is complete

Proof: (i)  $\rightarrow$  (ii)

Let  $(x_n)$  be a sequence in a compact Metric Space  $X$ .

Expect to be that, if no subsequence of  $(x_n)$  converges in  $X$ . for each  $x \in X$ ,

There is some  $r_x > 0$  and positive integer  $n_x$  to such an extent that  $x_n \notin U(x, r_x)$  for all  $n \geq n_x$

Since  $X = \bigcup \{U(x, r_x) : x \in X\}$  and  $X$  is compact. A subset of  $E$  of a Metric Space  $X$  is supposed to be compact if every cover of  $E$  by open subsets of  $X$  has a limited subcover.

. There are  $x_1, x_2, x_3, \dots, x_m \in X$  Such that

$$X = U(x_1, r_{x_1}) \cup \dots \cup U(x_m, r_{x_m})$$

But if  $n_0 = \max\{n_{x_1}, \dots, n_{x_m}\}$  then  $x_{n_0} \notin U(x_j, r_{x_j}) \quad j = 1, 2, \dots, m$

So that  $x_{n_0} \notin X$  from

(i) Which is an inconsistency Subsequently  $(x_n)$  has a convergent subsequence.

(ii)  $\rightarrow$  (iii)

Suppose that every sequence in  $X$  has a convergent subsequence. Since a Cauchy sequence having a convergent subsequence is itself convergent. Along these lines  $X$  is Complete. Expect that  $X$  isn't completely limited.

There is a few  $\epsilon > 0$  to such an extent that  $X$  can't be covered by limitedly many open bundles of range  $\epsilon$  Let  $x_1 \in X$ , consider  $x_1, x_2, \dots, x_n \in X$ . Find  $x_{n+1} \in U(x_1, \epsilon) \dots \cup U(x_n, \epsilon) \quad n = 1, 2, \dots$ . Then  $d(x_n, x_m) \geq \epsilon$  for all  $n, m = 1, 2, \dots$ . So that  $(x_n)$  can not have a focalized convergent subsequence. Which is an inconsistency Subsequently  $X$  is completely limited.

(iii)  $\rightarrow$  (i)

Expect that  $X$  is complete and completely limited.

Consider an open front of  $X$  with no limited subcover.



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Since X is completely limited,

Cover X by limitedly many open wads of sweep 1. Then for atleast one of these open balls of span 1 ..say  $U_0 = U(x_0, 1)$   
 " Subset of the completely limited measurement space X,  $U_0$  is itself completely limited."

there is some  $x_1 \in U_0$  to such an extent that

$$U_1 = U\left(x, \frac{1}{2}\right) \text{ has no limited subcover from the given open cover of } X$$

We discover a sequence  $(x_n)$  in X with  $x_{n+1} \in U_n = U\left(x_n, \frac{1}{2^n}\right)$

To such an extent that  $U_n$  has no limited subcover from the given open front of X.

$$d(x_n, x_m) \leq \sum_{j=n}^{m-1} d(x_n, x_m) \leq \sum_{j=n}^{m-1} \frac{1}{2^j} < \frac{1}{2^{n-1}} \quad \text{for all } m > n$$

$d(x_n, x_m)$  converges.

$(x_n)$  is a Cauchy sequence in the complete Metric Space X.

Let  $(x_n)$  is a Cauchy sequence in the complete measurement space X.

Let  $(x_n)$  converges to x in X

At that point  $x \in E$  (open arrangement) of the given front Of X.

Let  $r > 0$  be with the end goal that  $U = U(x, r) \subset E$

$$\text{pick } n, \text{ so huge } d(x_n, x) < \frac{r}{2}$$

$$\frac{1}{2^n} < \frac{r}{2}$$

On the off chance that  $y \in X$ , and  $d(x_n, y) < \frac{1}{2^n}$  then,

$$d(x, y) \leq d(x, x_n) + d(x_n, y)$$

$$< \frac{r}{2} + \frac{1}{2^n} < r$$

Consequently  $U_n \subset U \subset E$

{E} establishes a limited subcover for  $U_n$

Which is an inconsistency of  $U_n$

### Linear map from a normed space X to a normed space Y to be continuous

Let X and Y be linear space over K.

A linear map from X to Y is a function  $F : X \rightarrow Y$  such that

$$F(k_1x_1 + k_2x_2) = k_1 F(x_1) + k_2 F(x_2) \text{ for all } x_1, x_2 \in X, k_1, k_2 \in K$$

#### Range space of F.

$$R(F) = \{ y \in Y : F(x) = y \text{ for some } x \in X \}$$

Of Y is called the range space of F

#### Zero space of F.

The subspace  $Z(f) = \{ x \in X : F(x) = 0 \}$  of X is called the Zero space of F.

If  $Z(F) = X$  then  $F = 0$

#### Theorem 3.2:

Let X and Y be Normed Spaces and  $F : X \rightarrow Y$  be linear map the following conditions are equivalent. (i) F is limited on  $\bar{U}(0, 1)$  for some  $r > 0$

- (ii) F is continuous at 0
- (iii) F is continuous at X
- (iv) F is continuous is uniformly continuous on X
- (v)  $\|F(x)\| \leq \alpha \|x\|$  for all  $x \in X$  and some  $\alpha > 0$
- (vi) The Zero space  $Z(F)$  of F is closed in X. and the linear map  $\tilde{F} : X/Z(F) \rightarrow Y$  defined by  $\tilde{F}(x + Z(F)) = F(x)$ ,  $x \in X$  is continuous



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**Proof : (i) ⇒ (v)**

F is bounded on  $\bar{U}(0, 1)$  for some  $r > 0$

Then,  $\|F(x)\| \leq \beta \forall x \in \bar{U}(0, 1) \dots \dots \dots (1)$

Where  $\beta > 0$  is real number.

If  $r = 0$  then  $\|F(x)\| \leq \alpha \|x\|$  for all  $x \in X$  and some  $\alpha > 0$

$$\text{Let } x \in X, x \neq 0 \text{ Let } y = \frac{rx}{\|x\|}, \|y\| = \frac{|x|\|x\|}{\|x\|} = r$$

$$\Rightarrow y \in \bar{U}(0, 1) \text{ from (1) } \|F(y)\| \leq \beta$$

$$\left\| F\left(\frac{rx}{\|x\|}\right) \right\| \leq \beta \forall x \in X$$

$$\left\| \left(\frac{r}{\|x\|} F(x)\right) \right\| \leq \beta \forall x \in X$$

$$\frac{r}{\|x\|} \|F(x)\| \leq \beta \forall x \in X$$

$$\|F(x)\| \leq \frac{\beta}{r} \|x\|, \forall x \in X$$

Let  $\alpha = \frac{\beta}{r} > 0$ ,  $\|F(x)\| \leq \alpha \|x\|$  for all  $x \in X$  and some  $\alpha > 0$

**(v) ⇒ (iv)**

$$\|F(x)\| \leq \alpha \|x\| \text{ for all } x \in X \text{ and some } \alpha > 0$$

Let  $x, y \in X$ , and  $\epsilon > 0$  be given. Let  $\delta = \frac{\epsilon}{\alpha} > 0$  and  $z = x - y$

Since,  $z \in X$ ,  $\|F(z)\| \leq \alpha$

$$\|z\| \|F(x - y)\| \leq \alpha \|x - y\|$$

$$\|x - y\| < \delta, \Rightarrow \|F(x) - F(y)\| < \alpha \cdot \delta, \delta = \alpha \cdot \frac{\epsilon}{\alpha}$$

$$\Rightarrow \|F(x) - F(y)\| < \epsilon, \forall x \in X$$

F is uniformly continuous on X.

**(iv) ⇒ (iii)**

Every uniformly continuous function is continuous.

**(iii) ⇒ (iv)**

Since F is continuous on X, it is continuous at 0.

**(ii) ⇒ (i)**

Let F be continuous at 0. If given  $\epsilon > 0 \exists \delta = r > 0$  such that  $\|x - 0\| < r$

$$\Rightarrow \|F(x) - F(y)\| < \epsilon$$

$$x \in \bar{U}(0, 1) \Rightarrow \|F(x)\| < \epsilon$$

$$\|F(x)\| < \epsilon \forall x \in \bar{U}(0, 1)$$

Therefore F is limited on  $\bar{U}(0, 1)$

**(iii) ⇒ (v)**

Let F be a continuous on X, then the Zero space of F is Z (F)

$$Z(F) = \{x \in X / F(x) = 0\} = F^{-1}(\{0\})$$

Since  $\{0\}$  is closed in Y and  $F : X \rightarrow Y$  is continuous on X

$$F^{-1}(\{0\}) \text{ is closed in } X$$

Therefore Z (F) is closed in X.





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$X/Z(F)$  is quotient space with the norm.

$$\| \|x + Z(F)\| \| = \inf\{ \|x + Z(F)\| / Z \in Z(F) \} \text{-----(1)}$$

Define  $\tilde{F}: X/Z(F) \rightarrow Y$  by  $\tilde{F}(x + Z(F)) = F(x) \quad \forall x + Z(F) \in X/Z(F)$

Let  $x + Z(F), y + Z(F) \in X/Z(F)$  then  $x - y \in X/Z(F)$

$$F(x - y) = 0 \Rightarrow F(x) - F(y) = 0 \text{ (i.e.) } F(x) = F(y)$$

$$\tilde{F}(x + Z(F)) = \tilde{F}(y + Z(F))$$

$\tilde{F}$  is Well defined. Since F is continuous on X is True

F is continuous at 0

F is bounded on  $\bar{U}(0, 1)$  for some  $r > 0$

$$\|F(x)\| \leq \alpha \|x\| \text{ for all } x \in X \text{ and some } \alpha > 0 \text{ are true.}$$

Let  $x \in X$ , and  $z \in Z(F)$ , Then  $\| \tilde{F}(x + Z(F)) \| = \| \tilde{F}(x + Z + Z(F)) \| = \| F(x + Z) \| \leq \alpha \|x + Z\| \quad \forall z \in Z(F)$  by ... .. (2)

By taking infimum on both sides,

$$\frac{\inf_{z \in Z(F)} \| \tilde{F}(x + Z(F)) \|}{\inf_{z \in Z(F)} \|x + Z(F)\|} \leq \frac{\inf_{z \in Z(F)} \{ \alpha \|x + Z\| \}}{\inf_{z \in Z(F)} \|x + Z(F)\|} \text{ from (1)}$$

$$\| \tilde{F}(x + Z(F)) \| \leq \alpha \inf\{ \|x + Z\| / z \in Z(F) \} \\ \leq \alpha \|x + Z(F)\| \quad \forall x + Z(F) \in X/Z(F)$$

$\therefore$  (v) is true for  $\tilde{F}$  defined on  $X/Z(F)$

Since,  $\|F(x)\| \leq \alpha \|x\|$  for all  $x \in X$  and some  $\alpha > 0$

Therefore F is uniformly continuous on X

F is continuous on X

$$\therefore \tilde{F} \text{ is continuous on } X/Z(F) \text{ ... .. (3)}$$

(vi)  $\rightarrow$  (iii)

Let  $Z(F)$  be closed in X and  $\tilde{F}: X/Z(F) \rightarrow Y$  defined by  $\tilde{F}(x + Z(F)) = F(x)$

For every  $x + Z(F) \in X/Z(F)$  is continuous.

Since  $\tilde{F}$  is continuous on  $X/Z(F)$  ... .. by (3)

(iii)  $\Rightarrow$  (ii)  $\Rightarrow$  (i)  $\Rightarrow$  (v)

(v) is true for  $\tilde{F}$

$$\| \tilde{F}(x + Z(F)) \| \leq \alpha \|x + Z(F)\|, \text{ For every } x + Z(F) \in X/Z(F) \text{ and } \alpha > 0 \\ = \alpha \inf\{ \|x + Z\| / Z \in Z(F) \}, \forall x \in X, 0 \in Z(F) \\ \leq \alpha \|x + 0\|, \forall x \in X \text{ which is true for F}$$

$$\|F(x)\| \leq \alpha \|x\| \quad \forall x \in X \text{ which is true for F}$$

(v)  $\Rightarrow$  (iv)  $\Rightarrow$  (iii)  $\therefore$  F is continuous on X.

#### IV. CONCLUSION

In this work, we examined the fundamental and adequate conditions on normed Linear spaces, that guarantee the boundedness of any direct map. Also, we inferred the presence of fixed point hypotheses for Quotient space. Besides, a few models are given any place important. Our outcomes might be the inspiration to different creators for stretching out and improving these outcomes to be reasonable apparatuses for their applications.



**15<sup>th</sup> & 16<sup>th</sup> September 2021**

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**Acknowledgements:** The creator might want to offer her genuine thanks to Dr.P.Thamilselvi for his significant help and inspiration.

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