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Exact Solutions of KDV Equation with Forcing Term by Generalized Tanh-Coth and G/G' Expansion Method

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ABSTRACT: An application of the generalized tanh-coth method and the (G'/G) -expansion method to search for exact solutions of nonlinear partial differential equations is analyzed. These methods are used for the KdV equation with forcing term. The generalized tanh-coth method and the (G'/G) -expansion method were used to construct periodic wave and solitary wave solutions of nonlinear evolution equations. It is shown that the generalized tanh-coth method and the (G'/G) -expansion method, with the help of symbolic computation, provide a straightforward and powerful mathematical tool for solving nonlinear problems.

I. INTRODUCTION

Nonlinear phenomena play a fundamental role in applied mathematics and physics. Recently, the study of nonlinear partial differential equations in modelling physical phenomena has become an important tool. The investigation of the travelling wave solutions plays an important role in nonlinear sciences. A variety of powerful methods have been presented, such as the inverse scattering transform [1], Hirota's bilinear method [2], sine-cosine method [3], homotopy perturbation method [4], homotopy analysis method [5, 6], variational iteration method [7, 8], tanh-function method [9], Backlund transformation [10], Exp-function method [11–16], tanh-coth method [17–20], (G'/G) -expansion method [21–23], Laplace Adomian decomposition method [24], and differential transform method [25]. Here, we use an effective method for constructing a range of exact solutions for following nonlinear partial differential equations that in this paper we developed solutions as well. The standard tanh method is well-known analytical method which is first presented by Malfliet's [26] and developed in [26, 27]. In this paper, we explain method which is called the generalized tanh-coth method to look for travelling wave solutions of nonlinear evolution equations. The KdV equation with forcing term has been investigated by Zhang [28] is in the form

$$u_t + \alpha u u_x + \beta u_{xxx} = F(t), \quad (1)$$

where $F(t)$ external forcing function varying with time t , α and β are constants. Our aim of this paper is to obtain analytical solutions of the KdV equation with forcing term and to determine the accuracy of the aforementioned methods in solving these kinds of problems. Symmetries of differential equations are one of the most important concepts in theory of differential equations and physics. One of the most prominent equations is KdV (Korteweg-de Vries) equation with application in shallow water theory. There are diverse methods for finding symmetries of differential equations. One of the most important ones is Lie method. A symmetry is a mapping of one mathematical object into itself or into another mathematical object that preserves some property of the object.

Lie Group Symmetries. A symmetry is a mapping of one mathematical object into itself or into another mathematical object that preserves some property of the object. The easiest symmetries to see are the discrete symmetries of geometrical objects, such as the rotational symmetries of the objects. Note that the sphere in the middle is invariant under a continuous group of rotational symmetries, not just a discrete group. We have been speaking implicitly about groups of transformations of families of curve $\phi(x, y) = c$ in the plane. Let $x = (x, y)$ and $X = (X, Y)$ be points in the Euclidean plane, and for λ in \mathbb{R} , let $P_\lambda : x \rightarrow (x, \lambda) = X$ be a transformation, depending on the parameter λ that takes points x to points X . Usually, we would give a rule $\Psi(\mu, \nu)$ defining the composition of two parameters

μ and ν , but we can always reparameterize the group, so that the composition is additive; that is, $\Psi(\lambda_1, \lambda_2) = \lambda_1 + \lambda_2$. With this parameterization, the identity element λ_0 becomes $\lambda_0 = 0$. We say the set of transformation P_λ is a (additive) transformation group G if the following conditions are satisfied:

- (1) P_λ is one-to-one and onto;
- (2) $P_{\lambda_2} \circ P_{\lambda_1} = P_{\lambda_2 + \lambda_1}$; that is, $f(f(x, \lambda_1), \lambda_2) = f(x, \lambda_2 + \lambda_1)$;
- (3) $P_\lambda = I$ (i.e. $f(x, 0) = x$)
- (4) for each λ_1 there exists a unique $\lambda_2 = -\lambda_1$ such that $P_{\lambda_2} \circ P_{\lambda_1} = P_0 = I$; that is,

$$f(f(x, \lambda_1), \lambda_2) = f(x, 0) = x.$$

If, in addition to these four group properties, f is infinitely differentiable with respect to x and analytic with respect to λ , we say that G is a one-parameter Lie group (or a Lie point transformation) [29]. A point transformation maps points in the Euclidean plane into other points in the plane. There are more general transformations, such as contact transformations and Lie-Backlund transformations. The paper is organized as follows. In Sections 2 and 3, we briefly give the steps of the methods and apply the methods to solve the nonlinear partial differential equations. In Section 4, we examine KdV equation with forcing term using the before sections of methods. Also a conclusion is given in Section 5. Finally some references are given at the end of this paper.

II. BASIC IDEA OF THE (G'/G) -EXPANSION METHOD

Another powerful analytical method is called (G'/G) -expansion method; we give the detailed description of method which is first presented by Wang et al. [30].

Step 1. For given NLPDE with independent variables $X = (x, y, z, t)$ and dependent variable u

$$\mathcal{P}(u, u_t, u_x, u_y, u_z, u_{xx}, u_{yy}, u_{zz}, u_{xy}, u_{xt}, u_{ty}, u_{tz}, \dots) = 0 \quad (2)$$

Can be converted to on ODE

$$\mathcal{M}(u, -cu', u', u'', \dots) = 0, \quad (3)$$

at which transformation $\xi = x + y - ct$ is wave variable. Also c is constant to be determined later.

Step 2. We seek its solutions in the more general polynomial form as follows:

$$u(\xi) = a_0 + \sum_{k=1}^m a_k \left(\frac{G'(\xi)}{G(\xi)} \right)^k, \quad (4)$$

where $G(\xi)$ satisfies the second order LODE in the form

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \quad (5)$$

where $a_0, a_k (k=1, 2, \dots, m), \lambda$, and μ are constants to be determined later, $a_m \neq 0$, but the degree of which is generally equal to or less than $m-1$, and the positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (3).

Step 3. Substitute (4) and (5) into (3) with the value of n obtained in Step 1. Collecting the coefficients of $(G'(\xi)/G(\xi))^k (k=0, 1, 2, \dots)$ and then setting each coefficient to zero, we can get a set of overdetermined partial differential equations for $a_0, a_i (i=1, 2, \dots, n), \lambda, c$, and μ with the aid of symbolic computation Maple.

Step 4. Solving the algebraic equation in Step 3 and then substituting a_i, \dots, a_m, c and general solution of (5) into (4), we can obtain a series of fundamental solutions of (2) depending of the solution $G(\xi)$ of (5).

III. BASIC IDEA OF THE GENERALIZED TANH-COTH METHOD

We now describe the generalized tanh-coth method for the given partial differential equations. We give the detailed description of method which to use this method, we take following steps.

Step 1. For a given NLPDE with independent variable $X = (x, y, z, t)$ and dependent variable u , we consider a general form of nonlinear equation

$$\mathcal{P}(u, u_t, u_x, u_y, u_z, u_{xx}, u_{yy}, u_{zz}, u_{xy}, u_{xt}, u_{ty}, u_{tz}, \dots) = 0 \quad (6)$$

which can be converted to on ODE

$$\mathcal{Q}(u, -cu', u', u'', u''', \dots) = 0, \quad (7)$$

at which transformation $\xi = x + y - ct$ is wave variable. Also, c is constant to be determined later.

Step 2. We introduce the Riccati equation as follows:

$$\Phi' = r + p\Phi + q\Phi^2, \quad \Phi = \Phi(\xi), \quad \xi = x + y - ct, \quad (8)$$

that leads to the change of derivatives

$$\begin{aligned} \frac{d}{d\xi} &= (r + p\Phi + q\Phi^2) \frac{d}{d\Phi}, \\ \frac{d^2}{d\xi^2} &= (r + p\Phi + q\Phi^2) \times \left((p + 2q\Phi) \frac{d}{d\Phi} + (r + p\Phi + q\Phi^2) \frac{d^2}{d\Phi^2} \right), \\ \frac{d^3}{d\xi^3} &= (r + p\Phi + q\Phi^2) \times \left((6q^2\Phi^2 + 6pq\Phi + 2rq + p^2) \frac{d}{d\Phi} \right. \\ &\quad \left. + (6q^2\Phi^3 + 9pq\Phi^2 + 3(p^2 + 2rq)\Phi + 3rp) \right. \\ &\quad \left. \times \frac{d^2}{d\Phi^2} + (r + p\Phi + q\Phi^2)^2 \frac{d^3}{d\Phi^3} \right) \end{aligned} \quad (9)$$

which admits the use of a finite series of functions of the form

$$u(\xi) = S(\Phi) = \sum_{k=0}^m a_k \Phi^k + \sum_{k=1}^m b_k \Phi^{-k}, \quad (10)$$

where $a_k (k=0, 2, \dots, m), b_k (k=1, 2, \dots, m), p, r,$ and q are constant to be determined later. But the positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (7). If m is not an integer, then a transformation formula should be used to overcome this difficulty. For aforementioned method, expansion (10) reduces to the standard tanh method [26] for $b_k = 0, 1 \leq k \leq m$.

Step 3. Substitute (8) and (9) into (7) with the value of m obtained in Step 2. Collecting the coefficients of $\Phi^k (k=0, 1, 2, \dots)$ and then setting each coefficient to zero, we can get a set of overdetermined partial differential equations for $a_0, a_i (i=1, 2, \dots, m), b_i (i=1, 2, \dots, m), p, q,$ and r with the aid of symbolic computation Maple.

Step 4. Solving the algebraic equation in Step 3 and then substituting $a_0, a_1, b_1, \dots, a_m, b_m, c$ in (10).

We will consider the following special solutions of the Riccati equation (8)

Case 1. For each $p, r,$ and $q \neq 0$, (8) has the following solution:

$$\begin{aligned} \Phi(\xi) &= \frac{-p}{2q} + \frac{\sqrt{-\Delta}}{2q} \tan \left(\frac{\sqrt{-\Delta}\xi}{2} + C \right), \\ \Delta &= p^2 - 4qr, \quad \xi = kx + \int \tau(t) dt, \end{aligned} \quad (11)$$

$$\text{Or } \Phi(\xi) = \frac{-p}{2q} - \frac{\sqrt{\Delta}}{2q} \tanh\left(\frac{\sqrt{\Delta}\xi}{2} + C\right), \quad (12)$$

$$\Delta = p^2 - 4qr, \quad \xi = kx + \int \tau(t)dt,$$

$$\text{Or } \Phi(\xi) = \frac{-p}{2q} + \frac{\sqrt{-\Delta}}{2q} \cot\left(\frac{\sqrt{-\Delta}\xi}{2} + C\right), \quad (13)$$

$$\Delta = p^2 - 4qr, \quad \xi = kx + \int \tau(t)dt,$$

$$\text{Or } \Phi(\xi) = \frac{-p}{2q} + \frac{\sqrt{\Delta}}{2q} \coth\left(\frac{\sqrt{\Delta}\xi}{2} + C\right), \quad (14)$$

$$\Delta = p^2 - 4qr, \quad \xi = kx + \int \tau(t)dt,$$

where C is constant.

Case 2. For $p = r = 1$ and $q = 0$, (8) has the following solution

$$\Phi(\xi) = e^\xi - 1, \quad \xi = kx + \int \tau(t)dt. \quad (15)$$

Case 3. For $r = 1/2$, $p = 0$, and $q = -1/2$, (8) has the following solution:

$$\Phi(\xi) = -i \tan\left(\frac{\xi i}{2}\right), \quad \text{or} \quad \Phi(\xi) = \tanh\left(\frac{\xi i}{2}\right), \quad \xi = \xi = kx + \int \tau(t)dt, \quad (16)$$

$$\Phi(\xi) = -i \cot\left(\frac{\xi i}{2}\right), \quad \text{or} \quad \Phi(\xi) = \coth\left(\frac{\xi}{2}\right), \quad \xi = \xi = kx + \int \tau(t)dt, \quad (17)$$

$$\Phi(\xi) = -i \tan(i\xi) \pm i \sec(i\xi), \quad \text{Or } \Phi(\xi) = \tanh(\xi) \pm i \sec(\xi), \quad \xi = \xi = kx + \int \tau(t)dt, \quad (18)$$

$$\text{But, we know } \tanh\left(\frac{\xi}{2}\right) = \coth(\xi) - \operatorname{csch}(\xi), \quad \coth\left(\frac{\xi}{2}\right) = \coth(\xi) + \operatorname{csch}(\xi). \quad (19)$$

Case 4. For $r = 1$, $p = 1$, and $q = -1$, (8) has the following solution:

$$\begin{aligned} \Phi(\xi) &= \frac{1}{2} + \frac{\sqrt{5}}{2} \tanh\left(\frac{\sqrt{5}}{2} \xi\right), \\ \Phi(\xi) &= \frac{1}{2} - \frac{\sqrt{5}}{2} i \tan\left(\frac{\sqrt{5}}{2} i \xi\right), \end{aligned} \quad (20)$$

$$\Phi(\xi) = \frac{1}{2} + \frac{\sqrt{5}}{2} \coth\left(\frac{\sqrt{5}}{2} \xi\right),$$

$$\text{or } \Phi(\xi) = \frac{1}{2} + \frac{\sqrt{5}}{2} i \cot\left(\frac{\sqrt{5}}{2} i \xi\right).$$

IV. THE KDV EQUATION WITH FORCING TERM

Case I (application of the (G'/G) -expansion method). In this section, we employ the KdV equation with forcing term by using the (G'/G) -expansion method as follows:

$$u_t + \alpha u u_x + \beta u_{xxx} = F(t), \quad (21)$$

we suppose that

$$u(x, t) = v(x, t) + \int F(t) dt, \quad (22)$$

and then (21) reduces to

$$v_t + \alpha v v_x + \alpha \int F(t) dt v_x + \beta v_{xxx} = 0. \quad (23)$$

Using the transformation as follows

$$v = V(\mu), \quad \xi = kx + \int \tau(t) dt, \quad (24)$$

where k is a constant and $\tau(t)$ is a function of t to be determined later, (23) is carried to ODE

$$\tau(t) V' + k \alpha V V' + K \alpha \int F(t) dt V' + k^3 \beta V''' = 0, \quad (25)$$

$$\begin{aligned} V(\xi) &= a_m \left(\frac{G'(\xi)}{G(\xi)} \right)^m + \dots, \\ V_\xi(\xi) &= -m a_m \left(\frac{G'(\xi)}{G(\xi)} \right)^{m+1} + \dots, \\ V_{\xi\xi}(\xi) &= m(m+1) a_m \left(\frac{G'(\xi)}{G(\xi)} \right)^{m+2} + \dots, \\ V_{\xi\xi\xi}(\xi) &= -m(m+1)(m+2) a_m \left(\frac{G'(\xi)}{G(\xi)} \right)^{m+3} + \dots, \end{aligned} \quad (26)$$

where we conclude

$$m+3 = 2m+1, \quad (27)$$

and we find $m=2$. We can suppose that the solution of (21) is of the form

$$V(\xi) = a_0 + a_1 \left(\frac{G'(\xi)}{G(\xi)} \right) + a_2 \left(\frac{G'(\xi)}{G(\xi)} \right)^2, \quad a_2 \neq 0, \quad (28)$$

and therefore

$$\begin{aligned} V_\xi(\xi) &= -2a_2 \left(\frac{G'(\xi)}{G(\xi)} \right)^3 \\ &\quad - (a_1 + 2a_2 \lambda) \left(\frac{G'(\xi)}{G(\xi)} \right)^2 \\ &\quad - (2a_2 \mu + a_1 \lambda) \left(\frac{G'(\xi)}{G(\xi)} \right) - a_1 \mu \end{aligned}$$

$$V_{\xi\xi\xi}(\xi) = -24a_2 \left(\frac{G'(\xi)}{G(\xi)} \right)^5 - (6a_1 + 54a_2 \lambda) \left(\frac{G'(\xi)}{G(\xi)} \right)^4$$

$$\begin{aligned}
 & -(40a_2\mu + 12a_1\lambda + 38a_2\lambda^2) \left(\frac{G'(\xi)}{G(\xi)} \right)^3 \\
 & -(52a_2\lambda\mu + 8a_1\mu + 7a_1\lambda^2 + 8a_2\lambda^3) \left(\frac{G'(\xi)}{G(\xi)} \right)^2 \\
 & -(14a_2\lambda^2\mu + 8a_1\mu + a_1\lambda^3 + 16a_2\lambda^3) \left(\frac{G'(\xi)}{G(\xi)} \right) \\
 & -6a_2\lambda\mu^2 - a_1\lambda^2\mu - 2a_1\mu^2.
 \end{aligned} \tag{29}$$

Substituting (28) and (29) and by using the well-known software Maple, we obtain the system of following results:

$$a_0 = a_0, \quad a_1 = 0, \quad a_2 = -\frac{12k^2\beta}{\alpha}, \tag{30}$$

$$\tau(t) = -ka_0\alpha - k\alpha \int F(t)dt, \quad \lambda = 0, \quad \mu = 0,$$

$$\text{or } a_0 = a_0, \quad a_1 = -\frac{12k^2\beta\lambda}{\alpha}, \quad a_2 = -\frac{12k^2\beta}{\alpha}, \tag{31}$$

$$\lambda = \lambda, \quad \mu = 0, \quad \tau(t) = -ka_0\alpha - k^3\beta\lambda^2 - k\alpha \int F(t)dt,$$

$$\text{or } a_0 = a_0, \quad a_1 = -\frac{12k^2\beta\lambda}{\alpha}, \quad a_2 = -\frac{12k^2\beta}{\alpha}, \tag{32}$$

$$\lambda = \lambda, \quad \mu = 0, \quad \tau(t) = -ka_0\alpha - k^3\beta\lambda^2 - 8k^3\beta\mu - k\alpha \int F(t)dt,$$

where λ and μ are arbitrary constants. Substituting (30)–(32) into expression (28), can be written as

$$V(\xi) = a_0 - \frac{12k^2\beta}{\alpha} \left(\frac{G'(\xi)}{G(\xi)} \right)^2, \tag{33}$$

$$\xi = kx + \int \left(-ka_0\alpha - k\alpha \int F(t)dt \right) dt,$$

$$\text{or } V(\xi) = a_0 - \frac{12k^2\beta\lambda}{\alpha} \left(\frac{G'(\xi)}{G(\xi)} \right) - \frac{12k^2\beta}{\alpha} \left(\frac{G'(\xi)}{G(\xi)} \right)^2, \tag{34}$$

$$\xi = kx + \int \left(ka_0\alpha - k^3\beta\lambda^2 - k\alpha \int F(t)dt \right) dt,$$

$$\text{Or } V(\xi) = a_0 - \frac{12k^2\beta\lambda}{\alpha} \left(\frac{G'(\xi)}{G(\xi)} \right) - \frac{12k^2\beta}{\alpha} \left(\frac{G'(\xi)}{G(\xi)} \right)^2, \tag{35}$$

$$\xi = kx + \int \left(ka_0\alpha - k^3\beta\lambda^2 - 8k^3\beta\mu - k\alpha \int F(t)dt \right) dt,$$

Substituting the general solutions of (5) into (33)–(35), we have three types of exact solutions of (21) as follows.

$$\text{Case 1. } \lambda = \mu = 0 \text{ (33). Consider } u_1(\xi) = -\frac{12k^2\beta}{\alpha} \frac{C_2^2}{(C_1 + C_2\xi)^2} + a_0 + \int F(t)dt, \tag{36}$$

$$\xi = kx + \int \left(-ka_0\alpha - k\alpha \int F(t)dt \right) dt.$$

Case 2. $\lambda \neq 0, \mu = 0$ (34).

When $\lambda^2 - 4\mu > 0$, we obtain hyperbolic function solution

$$u_2(\xi) = -\frac{3k^2\beta}{\alpha}(\lambda^2 - 4\mu) \times \left(\left(C_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}\xi}{2}\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}\xi}{2}\right) \right) \times \left(C_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}\xi}{2}\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}\xi}{2}\right) \right)^{-1} \right)^2 - \frac{3k^2\beta\lambda^2}{\alpha} + a_0 + \int F(t)dt, \quad (37)$$

where $\xi = kx + \int (-ka_0\alpha - k^3\beta\lambda^2 - k\alpha \int F(t)dt)dt$. when $\lambda^2 - 4\mu = 0$, we get rational solution

$$u_3(\xi) = -\frac{12k^2\beta}{\alpha} \frac{C_2^2}{(C_1 + C_2\xi)^2} - \frac{3k^2\beta\lambda^2}{\alpha} + a_0 + \int F(t)dt, \quad (38)$$

where $\xi = kx + \int (-ka_0\alpha - k^3\beta\lambda^2 - k\alpha \int F(t)dt)dt$. If $C_1 \neq 0$, $C_2 = 0$, $\lambda > 0$, $\mu = 0$, then (37) gives

$$u_4(x,t) = -\frac{3k^2\beta}{\alpha}\lambda^2 \left(1 + \tanh^2 \frac{\lambda}{2} \times \left[kx + \int (-ka_0\alpha - k^3\beta\lambda^2 - k\alpha \int F(t)dt)dt \right] \right) + a_0 + \int F(t)dt. \quad (39)$$

In particular, if $C_2 \neq 0$, $C_1 = 0$, $\lambda > 0$, $\mu = 0$, then (37) gives

$$u_5(x,t) = -\frac{3k^2\beta}{\alpha}\lambda^2 \left(1 + \coth^2 \frac{\lambda}{2} \times \left[kx + \int (-ka_0\alpha - k^3\beta\lambda^2 - k\alpha \int F(t)dt)dt \right] \right) + a_0 + \int F(t)dt. \quad (40)$$

Case 3. $\lambda \neq 0$, $\mu \neq 0$ (35).

When $\lambda^2 - 4\mu > 0$, we obtain hyperbolic function solution

$$u_6(\xi) = -\frac{3k^2\beta}{\alpha}(\lambda^2 - 4\mu) \left(\left(C_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}\xi}{2}\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}\xi}{2}\right) \right) \times \left(C_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}\xi}{2}\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}\xi}{2}\right) \right)^{-1} \right)^2 - \frac{3k^2\beta\lambda^2}{\alpha} + a_0 + \int F(t)dt, \quad (41)$$

where $\xi = kx + \int (-la_0\alpha - k^3\beta\lambda^2 - 8k^3\beta\mu - k\alpha \int F(t)dt)dt$.

when $\lambda^2 - 4\mu < 0$, we have trigonometric function solution

$$u_7(\xi) = -\frac{3k^2\beta}{\alpha}(4\mu - \lambda^2) \times \left(\left(-C_1 \sin\left(\frac{\sqrt{4\mu - \lambda^2}\xi}{2}\right) + C_2 \cos\left(\frac{\sqrt{4\mu - \lambda^2}\xi}{2}\right) \right) \right) \quad (42)$$

$$\times \left(\left(-C_1 \cos \left(\frac{\sqrt{4\pi - \lambda^2 \xi}}{2} \right) + C_2 \sin \left(\frac{\sqrt{4\pi - \lambda^2 \xi}}{2} \right) \right)^{-1} \right)^2 - \frac{3k^2 \beta \lambda^2}{\alpha} + a_0 + \int F(t) dt,$$

where $\xi = kx + \int (-ka_0 \alpha - k^3 \beta \lambda^2 - 8k^3 \beta \mu - k\alpha) \int F(t) dt dt$.

where $\lambda^2 - 4\mu = 0$, we get rational solution

$$u_8(\xi) = -\frac{12k^2 \beta}{\alpha} \frac{C_2^2}{(C_1 + C_2 \xi)^2} - \frac{3k^2 \beta \lambda^2}{\alpha} + a_0 + \int F(t) dt, \quad (43)$$

where $\xi = kx + \int (-ka_0 \alpha - k^3 \beta \lambda^2 - 8k^3 \beta \mu - k\alpha) \int F(t) dt dt$.

If $C_1 \neq 0, C_2 = 0, \lambda > 0, \mu = 0$, then (41) gives

$$u_9(x, t) = -\frac{3k^2 \beta}{\alpha} \lambda^2 \left(1 + \tanh^2 \frac{\lambda}{2} \times \left[kx + \int (-ka_0 \alpha - k^3 \beta \lambda^2 - k\alpha) \int F(t) dt dt \right] \right) + a_0 + \int F(t) dt. \quad (44)$$

If $C_2 \neq 0, C_1 = 0, \lambda > 0, \mu = 0$, then (41) gives

$$u_{10}(x, t) = -\frac{3k^2 \beta}{\alpha} \lambda^2 \left(1 + \cot^2 \frac{\lambda}{2} \times \left[kx + \int (-ka_0 \alpha - k^3 \beta \lambda^2 - k\alpha) \int F(t) dt dt \right] \right) + a_0 + \int F(t) dt. \quad (45)$$

In particular, If $C_1 \neq 0, C_2 = 0, \lambda > 0, \mu = 0$, then (42) can be written as

$$u_{11}(x, t) = -\frac{12k^2 \beta}{\alpha} \mu \times \tan^2 \left(\sqrt{\mu} \left[kx + \int (-ka_0 \alpha - 8k^3 \beta \mu - k\alpha) \int F(t) dt dt \right] \right) + a_0 + \int F(t) dt. \quad (46)$$

And If $C_2 \neq 0, C_1 = 0, \lambda = 0, \mu > 0$, (42) can be written as

$$u_{12}(x, t) = -\frac{12k^2 \beta}{\alpha} \mu \times \cot^2 \left(\sqrt{\mu} \left[kx + \int (-ka_0 \alpha - 8k^3 \beta \mu - k\alpha) \int F(t) dt dt \right] \right) + a_0 + \int F(t) dt. \quad (47)$$

But if $C_1 \neq 0, C_2 = 0, \lambda = 0, \mu < 0$, then (41) gets

$$u_{13}(x, t) = -\frac{12k^2 \beta}{\alpha} \mu \times \tanh^2 \left(\sqrt{-\mu} \left[kx + \int (-ka_0 \alpha - 8k^3 \beta \mu - k\alpha) \int F(t) dt dt \right] \right) + a_0 + \int F(t) dt. \quad (48)$$

Also if $C_2 \neq 0, C_1 = 0, \lambda = 0, \mu < 0$, then (41) gets

$$u_{14}(x, t) = \frac{12k^2 \beta}{\alpha} \mu \times \coth^2 \left(\sqrt{-\mu} \left[kx + \int (-ka_0 \alpha - 8k^3 \beta \mu - k\alpha) \int F(t) dt dt \right] \right) + a_0 + \int F(t) dt. \quad (49)$$

For instance if $F(t) = \sin(t) + \cos(t), a_0 = \alpha = \beta = k = 1$, we get

$$\begin{aligned}
 u_9(x,t) &= -3\lambda^2 \left(1 + \tanh^2 \left(\frac{\lambda}{2} [x - (1 + \lambda^2)t - \sin(t) - \cos(t)] \right) \right) + 1 + \sin(t) - \cos(t), \\
 u_{10}(x,t) &= -3\lambda^2 \left(1 + \coth^2 \left(\frac{\lambda}{2} [x - (1 + \lambda^2)t - \sin(t) - \cos(t)] \right) \right) + 1 + \sin(t) - \cos(t), \\
 u_{11}(x,t) &= 12\mu \tan^2(\sqrt{\mu}[x - (1 - 8\mu)t - \sin(t) - \cos(t)]) + 1 + \sin(t) - \cos(t), \\
 u_{12}(x,t) &= 12\mu \cot^2(\sqrt{\mu}[x - (1 + 8\mu)t - \sin(t) - \cos(t)]) + 1 + \sin(t) - \cos(t), \\
 u_{13}(x,t) &= 12\mu \tanh^2(\sqrt{-\mu}[x - (1 + 8\mu)t - \sin(t) - \cos(t)]) + 1 + \sin(t) - \cos(t), \\
 u_{14}(x,t) &= 12\mu \coth^2(\sqrt{-\mu}[x - (1 + 8\mu)t - \sin(t) - \cos(t)]) + 1 + \sin(t) - \cos(t), \quad (50)
 \end{aligned}$$

which are the solitary wave solutions of the KdV equation with forcing term. It can be seen that the results are the same, with comparing results [28].

Case II (application of the generalized tanh-coth method). We now employ the KdV equation with forcing term by using the generalized tanh-coth method as follows:

$$u_t + \alpha u u_x + \beta u_{xxx} = F(t). \quad (51)$$

We suppose that $u(x,t) = v(x,t) + \int F(t)dt$, (52)

And then (21) reduces to $v_t + \alpha v v_x + \alpha \int F(t)dt v_x + \beta v_{xxx} = 0$. (53)

Using the transformation as follows $v = V(\mu)$, $\xi = kx + \int \tau(t)dt$, (54)

where k is a constant and $\tau(t)$ is a function of t to be determined later, (23) is carried to a ODE

$$\tau(t)V' + k\alpha VV' + k\alpha \int F(t)dt V' + k^3\beta V''' = 0, \quad (55)$$

where prime denotes the differential with respect to μ .

In order to determine value of m , we balance V''' with VV' in (55), and by using (10), we obtain $m = 2$. We can suppose that the solution of (51) is of the form

$$V(\xi) = a_0 + a_1\Phi + a_2\Phi^2 + \frac{b_1}{\Phi} + \frac{b_2}{\Phi^2} \quad (56)$$

Substituting (56) into (55) and using the well-known software Maple, we obtain the system of following results:

$$a_0 = a_0, \quad a_1 = 0, \quad a_2 = 0, \quad \tau(t) = -ka_0\alpha - k^3\beta p^2 - 8k^3\beta rq - k\alpha \int F(t)dt, \quad b_1 = -\frac{12\beta k^2 pr}{\alpha}; \quad b_2 = -\frac{12\beta k^2 r^2}{\alpha}, \quad (57)$$

or $a_0 = a_0, \quad a_1 = 0, \quad b_2 = 0, \quad \tau(t) = -ka_0\alpha - k^3\beta p^2 - 8k^3\beta rq - k\alpha \int F(t)dt$, (58)

$$a_1 = -\frac{12\beta k^2 pq}{\alpha}, \quad a_2 = -\frac{12\beta k^2 q^2}{\alpha},$$

where p, q, r , and a_0 are arbitrary constants, Substituting (57) and (58) into expression (56), we obtain

$$u(\xi) = a_0 - \frac{12\beta k^2 pr}{\alpha\Phi(\xi)} - \frac{12\beta k^2 r^2}{\alpha\Phi^2(\xi)} + \int F(t)dt, \quad (59)$$

where $\xi = kx - \int [ka_0\alpha + k^3\beta p^2 + 8k^3\beta rq + k\alpha \int F(t)dt]dt$ and

$$u(\xi) = a_0 - \frac{12\beta k^2 pr}{\alpha} \Phi(\xi) - \frac{12\beta k^2 q^2}{\alpha} \Phi^2(\xi) + \int F(t) dt, \quad (60)$$

where $\xi = kx - \int [ka_0\alpha + k^3\beta p^2 + 8k^3\beta r q + k\alpha] F(t) dt dt$

By the manipulation as explained in the previous section, we have the following.

(I) The First Set for (59). Using the relations (11)-(14), we have

$$\begin{aligned} u_1(\xi) &= a_0 - \frac{12\beta k^2}{\alpha} \times \left(\frac{r^2 + pr[-p/2q] + (\sqrt{-\Delta}/2q) \tan((\sqrt{-\Delta}\xi/2) + C)}{[(-p/2q) + (\sqrt{-\Delta}/2q) \tan((\sqrt{-\Delta}\xi/2) + C)]^2} \right) + \int F(t) dt, \\ u_2(\xi) &= a_0 - \frac{12\beta k^2}{\alpha} \times \left(\frac{r^2 + pr[(-p/2q) + (\sqrt{\Delta}/2q) \tanh((\sqrt{\Delta}\xi/2) + C)]}{[(-p/2q) + \sqrt{\Delta}/2q \tanh((\sqrt{\Delta}\xi/2) + C)]^2} \right) + \int F(t) dt, \\ u_3(\xi) &= a_0 - \frac{12\beta k^2}{\alpha} \times \left(\frac{r^2 + pr[(-p/2q) + (\sqrt{-\Delta}/2q) \cot((\sqrt{-\Delta}\xi/2) + C)]}{[(-p/2q) - (\sqrt{-\Delta}/2q) \cot((\sqrt{-\Delta}\xi/2) + C)]^2} \right) + \int F(t) dt, \\ u_4(\xi) &= a_0 - \frac{12\beta k^2}{\alpha} \times \left(\frac{r^2 + pr[-p/2q] + (\sqrt{-\Delta}/2q) \coth((\sqrt{-\Delta}\xi/2) + C)}{[(-p/2q) - (\sqrt{-\Delta}/2q) \cot((\sqrt{-\Delta}\xi/2) + C)]^2} \right) + \int F(t) dt, \end{aligned} \quad (61)$$

where $\xi = kx - \int [ka_0\alpha + k^3\beta p^2 + 8k^3\beta r q + k\alpha] F(t) dt dt$. By

using Case 2 in Section 3, relation $(15 u_4(\xi))$ gets

$$u_5(x, t) = a_0 - \frac{12\beta k^2}{\alpha} \times \left(\frac{e^{kx - \int [ka_0\alpha + k^3\beta + \int F(t) dt] dt}}{e^{kx - \int [ka_0\alpha + k^3\beta + k\alpha \int F(t) dt] dt} - 1} \right) + \int F(t) dt, \quad (62)$$

And by using Case 3 and using relations (18) and (19), we obtain

$$\begin{aligned} u_5(x, t) &= a_0 - \frac{12\beta k^2}{\alpha} \times \left(\frac{e^{kx - \int [ka_0\alpha + k^3\beta + \int F(t) dt] dt}}{e^{kx - \int [ka_0\alpha + k^3\beta + k\alpha \int F(t) dt] dt} - 1} \right) + \int F(t) dt, \\ u_6(\xi) &= a_0 - \frac{12\beta k^2}{\alpha} \left(\frac{r}{-i \tan(i\xi) \pm i \sec(i\xi)} \right)^2 + \int F(t) dt, \\ u_7(\xi) &= a_0 - \frac{12\beta k^2}{\alpha} \left(\frac{r}{\tanh(\xi) \pm i \sec(i\xi)} \right)^2 + \int F(t) dt, \\ u_8(\xi) &= a_0 - \frac{12\beta k^2}{\alpha} \left(\frac{r}{\coth(\xi) \pm \operatorname{csch}(\xi)} \right)^2 + \int F(t) dt, \end{aligned} \quad (63)$$

Where $\xi = kx - \int [ka_0\alpha - 2k^3\beta + k\alpha] F(t) dt dt$

Using Case 3 in Section 3 and using $\Phi = \tanh(\xi/2)$ or $\Phi = \coth(\xi/2)$, we get

$$u_9(x, t) = a_0 - \frac{12\beta k^2}{\alpha} \times \left(\frac{r}{\tanh((kx - \int [ka_0\alpha - 2k^3\beta + k\alpha \int f(t)dt]dt)/2)} \right)^2 + \int F(t)dt,$$

$$u_{10}(x, t) = a_0 - \frac{12\beta k^2}{\alpha} \times \left(\frac{r}{\coth((kx - \int [ka_0\alpha - 2k^3\beta + k\alpha \int F(t)dt]dt)/2)} \right)^2 + \int F(t)dt, \quad (64)$$

And by using Case 3 and using $\Phi = -i \tanh(i\xi/2)$ or $\Phi = -i \coth(i\xi/2)$, we get

$$u_{11}(x, t) = a_0 + \frac{12\beta k^2}{\alpha} \times \left(\frac{r}{\tan\left(\left(ikx - i \int [ka_0\alpha - 2k^3\beta + k\alpha \int F(t)dt]dt\right)/2\right)} \right)^2 + \int F(t)dt,$$

$$u_{12}(x, t) = a_0 + \frac{12\beta k^2}{\alpha} \times \left(\frac{r}{\cot\left(\left(ikx - i \int [ka_0\alpha - 2k^3\beta + k\alpha \int F(t)dt]dt\right)/2\right)} \right)^2 + \int F(t)dt, \quad (65)$$

And by using Case 4 in Section 3 and by using relations (20), we obtain

$$u_{13}(x, t) = a_0 + \int F(t)dt - \frac{12\beta k^2}{\alpha} \times$$

$$\left(\left(\frac{3}{2} + \frac{\sqrt{5}}{2} \tanh \frac{\sqrt{5}}{2} \times \left(kx - \int [ka_0\alpha + k^3\beta - 8k^3\beta + k\alpha \int F(t)dt]dt \right) \right) \right)$$

$$\times \left(\left[\frac{1}{2} + \frac{\sqrt{5}}{2} \tanh \frac{\sqrt{5}}{2} \times \left(kx - \int [ka_0\alpha + k^3\beta - 8k^3\beta + k\alpha \int F(t)dt]dt \right) \right]^2 \right)^{-1}.$$

$$u_{14}(x, t) = a_0 + \int F(t)dt - \frac{12\beta k^2}{\alpha}$$

$$\times \left(\left(\frac{3}{2} - \frac{\sqrt{5}}{2} i \tanh \frac{\sqrt{5}}{2} \times \left(ikx - i \int [ka_0\alpha + k^3\beta - 8k^3\beta + k\alpha \int F(t)dt]dt \right) \right) \right)$$

$$\times \left(\left[\frac{1}{2} - \frac{\sqrt{5}}{2} i \tanh \frac{\sqrt{5}}{2} \times \left(ikx - i \int [ka_0\alpha + k^3\beta - 8k^3\beta + k\alpha \int F(t)dt]dt \right) \right]^2 \right)^{-1}.$$

$$u_{15}(x, t) = a_0 + \int F(t)dt - \frac{12\beta k^2}{\alpha} \times$$

$$\left(\left(\frac{3}{2} + \frac{\sqrt{5}}{2} \coth \frac{\sqrt{5}}{2} \times \left(kx - \int [ka_0\alpha + k^3\beta - 8k^3\beta + k\alpha \int F(t)dt]dt \right) \right) \right)$$

$$\times \left(\left[\frac{1}{2} - \frac{\sqrt{5}}{2} \coth \frac{\sqrt{5}}{2} \times \left(kx - \int [ka_0\alpha + k^3\beta - 8k^3\beta + k\alpha \int F(t)dt]dt \right) \right]^2 \right)^{-1}.$$

$$u_{16}(x, t) = a_0 + \int F(t) dt - \frac{12\beta k^2}{\alpha} \times \left(\left(\frac{3}{2} + \frac{\sqrt{5}}{2} i \cot \frac{\sqrt{5}}{2} \times \left(ikx - \int \left[ka_0 \alpha + k^3 \beta - 8k^3 \beta + k\alpha \times \int F(t) dt \right] dt \right) \right) \times \left(\left[\frac{1}{2} + \frac{\sqrt{5}}{2} i \cot \frac{\sqrt{5}}{2} \times \left(ikx - i \int \left[ka_0 \alpha + k^3 \beta - 8k^3 \beta + k\alpha \times \int F(t) dt \right] dt \right) \right]^2 \right)^{-1} \right). \quad (66)$$

(II) The Second Set for (60). Using relations (11)-(14), we have

$$u_{17}(\xi) = a_0 - \frac{6\beta k^2 p}{\alpha} \left(-p + \sqrt{-\Delta} \tan \left[\frac{\sqrt{-\Delta} \xi}{2} + C \right] \right) - \frac{3\beta k^2}{\alpha} \left(-p + \sqrt{-\Delta} \tan \left[\frac{\sqrt{-\Delta} \xi}{2} + C \right] \right)^2 + \int F(t) dt,$$

$$u_{18}(\xi) = a_0 + \frac{6\beta k^2 p}{\alpha} \left(p + \sqrt{-\Delta} \tanh \left[\frac{\sqrt{\Delta} \xi}{2} + C \right] \right) - \frac{3\beta k^2}{\alpha} \left(p + \sqrt{-\Delta} \tanh \left[\frac{\sqrt{\Delta} \xi}{2} + C \right] \right)^2 + \int F(t) dt,$$

$$u_{19}(\xi) = a_0 + \frac{6\beta k^2 p}{\alpha} \left(p + \sqrt{-\Delta} \cot \left(\frac{\sqrt{-\Delta} \xi}{2} + C \right) \right) - \frac{3\beta k^2}{\alpha} \left(p + \sqrt{-\Delta} \cot \left(\frac{\sqrt{-\Delta} \xi}{2} + C \right) \right)^2 + \int F(t) dt,$$

$$u_{20}(\xi) = a_0 + \frac{6\beta k^2 p}{\alpha} \left(p + \sqrt{\Delta} \coth \left[\frac{\sqrt{\Delta} \xi}{2} + C \right] \right) - \frac{3\beta k^2}{\alpha} \left(p + \sqrt{\Delta} \coth \left[\frac{\sqrt{\Delta} \xi}{2} + C \right] \right)^2 + \int F(t) dt, \quad (67)$$

where $\xi = kx - \int [ka_0 \alpha + k^3 \beta p^2 + 8k^3 \beta r q + k\alpha \int F(t) dt] dt$. By using Case 3 and using relations (18) and (19), we obtain

$$u_{21}(\xi) = a_0 - \frac{3\beta k^2}{\alpha} (-i \tan(i\xi) \pm i \sec(i\xi))^2 + \int F(t) dt,$$

$$u_{22}(\xi) = a_0 - \frac{3\beta k^2}{\alpha} (\tanh(\xi) \pm i \sec(i\xi))^2 + \int F(t) dt,$$

$$u_{23}(\xi) = a_0 - \frac{3\beta k^2}{\alpha} (\coth(\xi) \pm \operatorname{csch}(\xi))^2 + \int F(t) dt, \quad (68)$$

where $\xi = kx - \int [ka_0 \alpha + 2k^3 \beta + k\alpha \int F(t) dt] dt$.

Using Case 3 in Section 3 and using $\Phi = \tanh(\xi/2)$ or $\Phi = \coth(\xi/2)$, we get

$$u_{24}(\xi) = a_0 - \frac{3\beta k^2}{\alpha} \times \tanh^2 \left(\frac{kx - \int \left[ka_0\alpha - 2k^3\beta + k\alpha \int F(t)dt \right] dt}{2} \right) + \int F(t)dt,$$

$$u_{25}(\xi) = a_0 - \frac{3\beta k^2}{\alpha} \times \coth^2 \left(\frac{kx - \int \left[ka_0\alpha - 2k^3\beta + k\alpha \int F(t)dt \right] dt}{2} \right) + \int F(t)dt, \quad (69)$$

And by using Case 3 and using $\Phi = i \tan(i\xi/2)$ or $\Phi = i \coth(i\xi/2)$, we get

$$u_{26}(\xi) = a_0 + \frac{3\beta k^2}{\alpha} \times \tan^2 \left(\frac{ikx - i \int \left[ka_0\alpha - 2k^3\beta + k\alpha \int F(t)dt \right] dt}{2} \right) + \int F(t)dt,$$

$$u_{27}(\xi) = a_0 + \frac{3\beta k^2}{\alpha} \times \cot^2 \left(\frac{ikx - i \int \left[ka_0\alpha - 2k^3\beta + k\alpha \int F(t)dt \right] dt}{2} \right) + \int F(t)dt, \quad (70)$$

And by using Case 4 in Section 3 and by using relation (20) we obtain

$$u_{28}(\xi) = a_0 + \frac{6\beta k^2}{\alpha} \left(1 + \sqrt{5} \tanh \left[\frac{\sqrt{5}}{2} \xi \right] \right) - \frac{3\beta k^2}{\alpha} \left(-\sqrt{5} \tanh \left[\frac{\sqrt{5}}{2} \xi \right] \right)^2 + \int F(t)dt,$$

$$u_{29}(\xi) = a_0 + \frac{6\beta k^2}{\alpha} \left(1 - \sqrt{5}i \tanh \left[\frac{\sqrt{5}}{2} i\xi \right] \right) - \frac{3\beta k^2}{\alpha} \left(1 - \sqrt{5}i \tanh \left[\frac{\sqrt{5}}{2} i\xi \right] \right)^2 + \int F(t)dt,$$

$$u_{30}(\xi) = a_0 + \frac{6\beta k^2}{\alpha} \left(1 + \sqrt{5} \coth \left[\frac{\sqrt{5}}{2} \xi \right] \right) - \frac{3\beta k^2}{\alpha} \left(1 + \sqrt{5} \coth \left[\frac{\sqrt{5}}{2} \xi \right] \right)^2 + \int F(t)dt,$$

$$u_{31}(\xi) = a_0 + \frac{6\beta k^2}{\alpha} \left(1 + \sqrt{5}i \cot \left[\frac{\sqrt{5}}{2} i\xi \right] \right) - \frac{3\beta k^2}{\alpha} \left(1 + \sqrt{5}i \cot \left[\frac{\sqrt{5}}{2} i\xi \right] \right)^2 + \int F(t)dt, \quad (71)$$

Where $\xi = kx - \int \left[ka_0\alpha + k^3\beta - 8k^3\beta + k\alpha \int F(t)dt \right] dt$. And above solutions are the exact solutions of the KdV equation with forcing term. We obtain solitary wave and periodic wave solution for the KdV equation with forcing term. It can be seen that the results are the same, with comparing results [28].

V. CONCLUSION

In this paper, we obtained exact solutions for the KdV equation with forcing term by using the generalized tanh-coth and (G'/G) -expansion methods. Generalized tanh-coth (G'/G) -expansion methods are useful methods for finding travelling wave solutions of nonlinear evolution equations. These methods have been successfully applied to obtain some new solitary wave and periodic wave solutions to the KdV equation with forcing term. Both generalized tanh-coth method and coth and (G'/G) -expansion method are more powerful in searching for exact solutions of NLPDEs. Some of these results are in agreement with the results reported in the literature. Comparing our results and Zhang's [28] results, then it can be seen that the results are the same. Also, new results are formally developed in this paper. It can be concluded that this method is a very powerful and efficient technique in finding exact solutions for wide classes of problems.



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