



# Mathematical Model of Two Layered Herschel Bulkley Fluid Model for Steady Flow of Blood under Velocity Slip

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**ABSTRACT:** Theoretical modeling in blood flow through narrow tubes could play an important role in the fundamental understanding, diagnosis and treatment of many cardiovascular diseases and arterial disorders. The constitution of blood demands a yield stress fluid model, and among the available yield stress fluid models for blood flow, the Herschel-Bulkley model is preferred (because Bingham, Power-law and Newtonian models are its special cases). In the constitutive equation of such a fluid there is one more additional parameter than Casson equation and Power-law fluid. In order to include the effects of yield stress as well as Power-law behavior in blood flow through narrow tubes, the present analysis deals with a two-layered flow in which core region is represented by a Herschel-Bulkley fluid and Peripheral region by a Newtonian fluid. In this model, a slip condition is introduced at fluids interface with the usual no-slip condition at blood vessel wall of five different locations of CVS. The expressions for velocity, plug flow velocity, wall shear stress, and the flux flow rate are derived. The results obtained for the flow characteristics reveal many interesting behaviors that warrant further study on the non-Newtonian fluid flow phenomena, especially the shear-thinning phenomena. Shear thinning reduces the wall shear stress.

**KEYWORDS:** Herschel-Bulkley fluid, Newtonian fluid, shearing stress, Non-Newtonian fluid.

## I. INTRODUCTION

Theoretical modeling in blood flow through narrow tubes could play an important role in the fundamental understanding, diagnosis and treatment of many cardiovascular diseases and arterial disorders. Blood being a suspension of cells in a continuous liquid base plasma, is a highly viscous fluid. Many investigators (Chaturani and Kaloni, 1976; Chaturani and Upadhyaya, 1979; Shukla et al., 1980; Majhi and Usha, 1984; Chaturani and Biswas, 1983; Philip and Chandra, 1996) have theoretically studied the flow of blood through uniform and stenosed tubes and analyzed the influence of slip velocity or peripheral plasma layer thickness on the flow variables such as velocity, wall shear stress and flow resistance. In these models, the peripheral layer thickness and slip velocity are assumed a priori based upon the experimental observations. To understand the flow patterns in stenosed arteries, Young (1968), Macdonald (1979), Deshpande et al. (1979), Shankar and Hemalatha (2006) etc. have analyzed the flow of blood through an arterial stenosis. Shukla et al. (1980) have taken two-layered models and analyzed the influence of peripheral plasma viscosity on flow characteristics. Chaturani and Kaloni (1976), Chaturani and Ponalagusamy (1982), Sankar and Lee (2007), Sankar and Ismail (2009), Sankar and Lee (2009) and Ponalagusamy (1986) have considered the flow of blood represented by a two-layered model. In their models, the peripheral layer thickness is assumed a priori. Biswas and Chakraborty (2010) have considered the flow of blood represented by a two-layered pulsatile blood flow in a stenosed artery with body acceleration and slip at wall. They studied that velocity and flow rate would increase but effective viscosity decreases due to wall slip model. In the works of Chaturani and Biswas (1983); Misra and Kar (1989), a velocity slip at the constricted wall is adopted. Also Nath (2003) considered a slip condition at the stenotic wall. Chaturani and Narasimman have considered blood to behave like a Herschel-Bulkley fluid. In the constitutive equation of such a fluid there is one more additional parameter than Casson equation and power law fluid. In order to include the effects of yield stress as well as power-law fluid behavior in blood flow through narrow tubes, the present analysis deals with a two-layered flow in which core region is represented by a Herschel-Bulkley fluid and peripheral region by a Newtonian fluid.

In the theoretical modeling on blood flow, different types of boundary conditions have been employed from various flow situations. A usual zero-slip condition for velocity is considered in the models Chaturani and Upadhyaya, Sing and Ram. From the above conditions, in this blood flow model, a slip condition is introduced at fluids interface with the usual no-slip condition at blood vessel wall of five different locations of cardiovascular system.



## II. BASIC EQUATIONS OF THE FLUID FLOW

Equations governing the flow are conservation equations of mass and momentum and constitutive equation of the fluid considered. The conservation of mass has the usual form and is expressed by the continuity equation

$$\dot{\rho} + \rho \vec{\nabla} \cdot \vec{v} = 0 \quad (1)$$

Where  $\rho$  is the density of the fluid,  $\vec{\nabla}$  the vector differential operator,  $\vec{v}$  the usual vel and the superim indicates the material time. The conservation omomentum has the form

$$T_{ij,j} + \rho b_i = \rho \dot{V}_i \quad (2)$$

Which is the equation of motion, where T is the stress tensot and  $b_i$  is tce per unit mass. For an incompressible fluid, equation of continuity (1) reduces to the form

$$\vec{\nabla} \cdot \vec{v} = 0 \quad (3)$$

And as result of equation (2-3), equation of motion for steady flow in absence of body force and interia terms, takes the form

$$\mu \vec{\nabla}^2 \vec{v} = \vec{\nabla} \rho \quad (4)$$

Where  $\mu$  is the fluid viscosity and  $\rho$  the pressure.

The constitutive equation of Herschel-Bulkley fluid is in the form

$$\tau = \tau_H + (K \dot{\gamma})^n \quad (5)$$

Where  $\tau_H$  is the Yield stress,  $\tau$  the shear stress,  $\dot{\gamma}$  shear rate, K and n are consistency and flow behavior indicates respectively that represent the non0Newtonian effect of such a fluid.

### I. Formulation of the Problem:

Steady laminar flow through a straight rigid tube of circular cross section is considered. Flowing fluid blood which is an incompressible one, has been assumed to behave like Herschel-Bulkley fluid. Flow is fully developed, 1-dimensional and free from effects. Let z-axis be chosen in the direction of flow along the axis of the pipe and r be taken along the radial direction, measured outwards from z-axis. If  $u_r$ ,  $u_\theta$  and  $u_z$  be the velocity components i.e  $\vec{V} = (u_r, u_\theta, u_z)$  in radial (r), tangential ( $\theta$ ) and axial (z) directions repectively then for one dimensional motion,  $u_r = 0 = u_\theta = 0$  and axial velocity  $u_z$  will be indepent of  $\theta$  and z. Equations of motion and continuity for fully developed steady viscous incompressible laminar flow in (r,  $\theta$ , z) system whose origin lies on the axis of the vessels are

$$0 = -\frac{\partial p}{\partial z} + \frac{1}{r} \frac{\partial}{\partial z} (r \tau_{rz}) \quad (6)$$

$$0 = \frac{\partial p}{\partial z} \quad (7)$$

$$0 = \frac{1}{r} \frac{\partial p}{\partial \theta} \quad (8)$$



And 
$$\frac{\partial u_z}{\partial r} = 0 \tag{9}$$

wherefrom we obtain  $u_r = u_r(z)$  and p is independent of r and  $\theta$ . Since velocity  $u_z$  along the axial direction, is a function of r only and the pressure p is independent of r so pressure gradient  $\frac{dp}{dz}$  must be a constant.

Equation (6) can be rewritten as

$$C + \frac{1}{r} \frac{\partial}{\partial r} (r\tau_{rz}) = 0 \tag{10}$$

where  $C = \left(-\frac{dp}{dz}\right)$  and shear stress component  $\tau_{rz}$ , for Newtonian fluid is  $\tau_{rz} = \mu \frac{\partial u_z}{\partial r}$  where  $\mu$  is shear viscosity and  $\tau_{rz}$  for Herschel-Bulkley fluid obtained from (5) is as follows:

$$-\frac{\partial u_z}{\partial r} = f(\tau) = \frac{1}{k} (r\tau_{rz} - \tau_H)^n, r\tau_{rz} \geq \tau_H \tag{11}$$

And  $\frac{\partial u_z}{\partial r} = 0$  for  $\tau_{rz} \leq \tau_H$  (12)

Where k and n are flow parameters which represents non-Newtonian effects ( $n < 1$  and positive),  $\tau_H$  the yield stress, z and r are axial and radial co-ordinates and  $u_z$  is the axial velocity along z-direction. The above relations corresponds to the vanishing of the velocity gradient in regions in which the shear stress  $\tau_{rz}$  is less than the yield stress  $\tau_H$

Equation of motion for one dimensional two-layered tube flow of Herschel-Bulkley fluid, for the core region ( $0 \leq r \leq a$ )

$$-\frac{\partial u_2}{\partial r} = \frac{1}{k} (\tau_{rz} - \tau_H)^n, r\tau_{rz} \geq \tau_H \tag{13}$$

And  $\frac{\partial u_2}{\partial r} = 0, \tau_{rz} \leq \tau_H$  (14)

And for peripheral region of thickness  $\delta$

$$c + \frac{1}{r} \frac{d}{dr} \left( r\mu_1 \frac{du_1}{dr} \right) = 0, a \leq r \leq R \tag{15}$$

Where  $u_1$  and  $u_2$  are axial velocities in the core and peripheral regions respectively and

$$\tau_{rz} = \left(\frac{r}{2}\right) \frac{dp}{dz}, \tau_w = \left(\frac{R}{2}\right) \frac{dp}{dz}, \tau_a = \left(\frac{a}{2}\right) \frac{dp}{dz}, \tau_H = \left(\frac{\tau_p}{2}\right) \frac{dp}{dz}, \beta = \frac{\tau_H}{\tau_w} = \frac{r_p}{R}, \beta_1 = \frac{\tau_H}{\tau_a} = \frac{r_p}{a} \tag{16}$$

### III. SOLUTIONS

Integrating equations (13--15) we get:

$$u_2(r) = c_1 - \frac{\frac{2}{dp}}{(n+1)k} \left( \frac{r}{2} \frac{dp}{dz} - \tau_H \right)^{n+1}, \tau_p \leq r \leq a \quad (17)$$

$$u_3(r) = 0, 0 \leq a \leq r_p \quad (18)$$

$$\text{And } u_1(r) = c_3 + c_4 l_n r - \frac{cr^2}{4\mu_1}, a \leq r \leq R \quad (19)$$

Where the constants  $C_i$  ( $i=1-4$ ) are to be determined from the following conditions and  $u_3(r)=u_2(r)$ .

### IV. BOUNDARY CONDITIONS

To obtained the constants  $C_i$  we have employed the following boundary conditions:

- i.  $u_1=0$  at  $r=0$
- ii.  $u_2=u_3$  at  $r=r_p$  (Plug core radius)
- iii. no-slip velocity at the vessel wall i.e.  $u_1=0$  at  $r=R$
- iv. slip condition is introduced at the interface of fluids:  $u_2-u_1=u_s$  at  $r=a$  (interface)
- v. core velocity is zero at the tube axis i.e.  $\frac{\partial u_2}{\partial r} = 0$  at  $r=0$

where  $u_s$  is an apparent slip in axial velocity.

Applying the above boundary conditions in equations (17)-(19), constants are evaluated as

$$c_4=0, c_3 = \frac{cR^2}{4\mu_1}, c_2 = u_p \text{ and } c_1 = u_s + \frac{2}{(n+1)k} \frac{dp}{dz} \left( \frac{a}{2} \frac{dp}{dz} - \tau_H \right)^{n+1} + \frac{c}{4\mu_1} (R^2 - a^2) \quad (20)$$

Where  $u_p$  is the plug core velocity. Using the above constants in equations (17) –(19), we can obtain the expressions of velocity as

$$u_1 = \frac{c}{4\mu_1} (R^2 - a^2), a \leq r \leq R \quad (21)$$

$$u_2 = u_s + \left( \frac{2}{(n+1)k} \frac{dp}{dz} \right) \left( \frac{a}{2} \frac{dp}{dz} - \tau_H \right)^{n+1} - \left( \frac{r}{2} \frac{dp}{dz} - \tau_H \right)^{n+1} + \frac{c}{4\mu_1} (R^2 - a^2), \tau_p \leq r \leq a \quad (22)$$

$$\text{And } u_2 = u_p, 0 \leq r \leq r_p \text{ where } c_p = u_s + \left( \frac{2}{(n+1)k} \frac{dp}{dz} \right) \left( \frac{a}{2} \frac{dp}{dz} - \tau_H \right)^{n+1} + \frac{c}{4\mu_1} (R^2 - a^2) \quad (23)$$



Velocity function can be rewritten in the form:

$$u_1 = \frac{c}{4\mu_1} \left( 1 - \left( \frac{r}{R} \right)^2 \right), a \leq r \leq R \quad (24)$$

$$u_2(r) = u_s + \frac{a\tau_a^n}{(n+1)k} \left[ (1 - \beta_1)^{n+1} - \left( \frac{r}{a} - \mu_1 \right)^{n+1} \right] + \frac{c}{4\mu_1} (R^2 - a^2), r_p \leq r \leq a \quad (25)$$

And  $u_2(r) = u_p, 0 \leq r \leq r_p$

Where  $u_p = u_s + \frac{a\tau_a^n}{(n+1)k} (1 - \beta_1)^{n+1} + \frac{c}{4\mu_1} (R^2 - a^2)$

Volumetric flow rate Q can be expressed from the formula  $Q = \int_{r=0}^R 2\pi r u(r) dr$  as follows

$$Q = (\pi a^2 u_s) + \frac{\pi a^3 \tau_a^n}{(n+1)k} (1 - \beta_1)^{n+1} \left\{ 1 - \frac{2}{(n+2)} (1 - \beta_1) - \frac{2}{(n+2)(n+3)} (1 - \beta_1)^2 \right\} + \frac{\pi c}{8\mu_1} (R^4 - a^4)$$

Or  $Q = (\pi a^2 u_s) + \frac{\pi a^3 \tau_a^n}{(n+3)k} (1 - \beta_1)^{n+1} \left\{ 1 + \frac{2}{(n+2)} \beta_1 + \frac{2}{(n+2)(n+1)} \beta_1^2 \right\} + \frac{\pi c}{8\mu_1} (R^4 - a^4)$  (25)

Apparent viscosity  $\phi = \frac{1}{\eta_a}$  where  $\frac{1}{\eta_a} = \frac{8Q}{\left( -\pi \frac{dp}{dz} R^4 \right)}$

has the expression

$$\phi = \frac{8u_s}{cR^2} \left( \frac{a}{R} \right)^2 + \frac{8\tau_a^n}{cR(n+3)k} \left( \frac{a}{R} \right)^3 (1 - \beta_1)^{n+1} A(\beta_1) + \frac{1}{\mu_1} \left[ 1 - \left( \frac{a}{R} \right)^4 \right] \quad (26)$$

Where  $A(\beta_1) = 1 + \frac{2}{(n+2)} \beta_1 + \frac{2}{(n+1)(n+2)} \beta_1^2$  and  $c = -\left( \frac{dp}{dz} \right)$

Expression for mean velocity of flow  $u_m$  can be obtained with the help of following formula

$$u_m = \frac{Q}{(\pi R^2)}$$

In the form  $u_m = \left( \frac{a}{R} \right)^2 u_s + \frac{a\tau_a^n}{(n+3)k} \left( \frac{a}{R} \right)^2 (1 - \beta_1)^{n+1} A(\beta_1) + \left( \frac{cR^2}{8\mu_1} \right) [1 - \left( \frac{a}{R} \right)^4]$  (27)

Expression for apparent viscosity  $\mu_a$  has the form

$$\mu_a = \phi^{-1} = \left[ \left( \frac{8u_s}{CR^2} \right) \left( \frac{a}{R} \right)^2 + \frac{8\tau_a^n}{cR(n+3)k} \left( \frac{a}{R} \right)^3 (1-\beta_1)^{n+1} .A(\beta_1) + \frac{1}{\mu_1} \left\{ 1 - \left( \frac{a}{R} \right)^4 \right\} \right]^{-1} \quad (28)$$

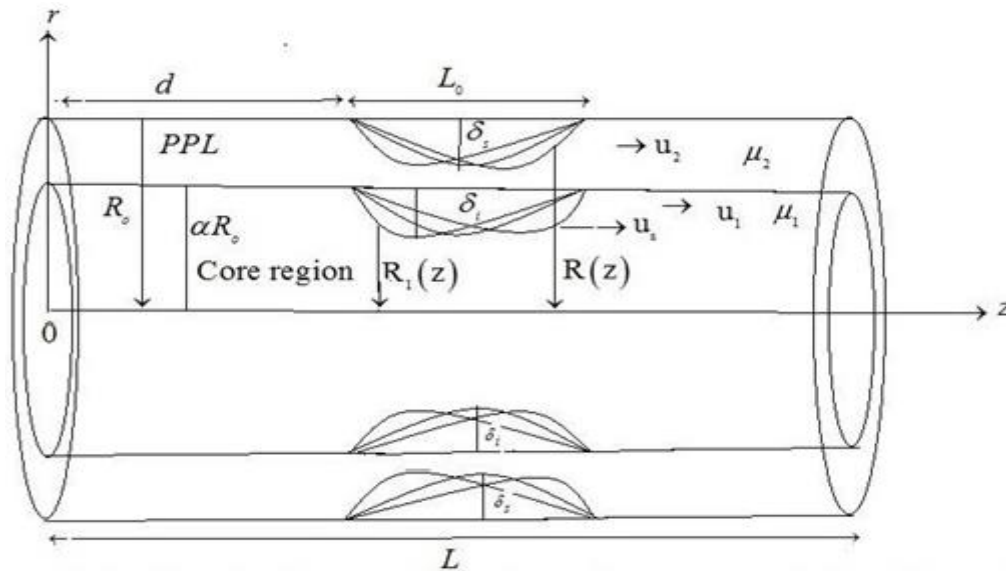


Fig. 1: Schematic diagram of Two-Layered Newtonian Model of Blood Flow Inside an Asymmetric Stenosed Artery

## V. RESULTS AND DISCUSSIONS

Blood flow through a uniform artery at different locations has been dealt with. Here blood being assumed to behave like a Herschel-Bulkley fluid and peripheral plasma region is represented by a Newtonian fluid [fig. 1], shear stress  $\tau_{rz}$  against rate of strain relationship exhibits from usual linear behavior. In addition to a finite yield stress  $\tau_H$ , this fluid is characterized by two flow parameters viz., fluid consistency  $k$  and behavior index  $n$ . As a result, for blood flow when  $r_p < R$ , there arises two regions viz.,  $0 \leq r \leq r_p$  and  $r_p \leq r \leq R$  and three separate cases viz., (i) in  $n < 1$ , fluid behaves as a pseudo plastic or a shear-thinning liquid, (ii) if  $n > 1$ , fluid behaves like a dilatants or a shear-thickening liquid and (iii) if  $n = 1$ , consistency curve being linear, fluid behaves as a Newtonian fluid with a constant viscosity. In computation of flow variables, magnitudes of pressure gradient and tube radius are used from Sud and Sekhon [ ] and  $n <, +, >$  for  $\tau_H \neq, \geq 0$ ,  $k = 2cp$  and slip velocity  $u_s = 0.1$  cm/sec are considered. As expected, velocity and flow rate increase but apparent viscosity decreases due to an axial velocity slip. Velocity profiles indicate a distinct behavior for variation of fluid property  $n$  and plugged radius  $r_p$

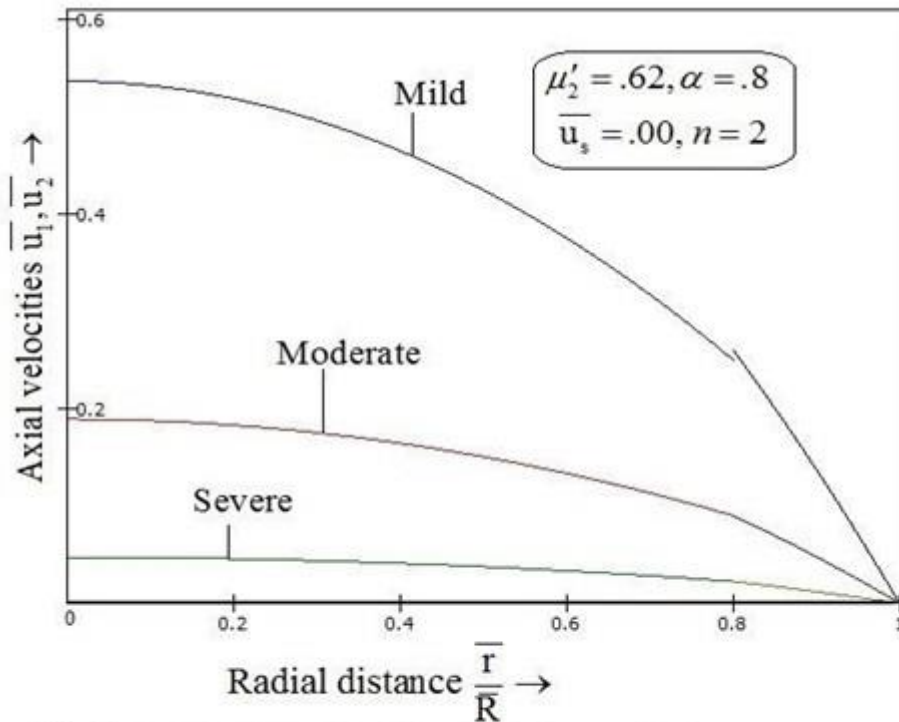


Fig.3. :Variation of axial velocities against radial distance and slip velocity  $\bar{u}_s = .00, n = 2, \mu_2' = .62$

At  $r_p=0$ , profiles indicate a parabolic trend in all vessels except arterioles and for  $r_p>0$ , profiles clearly indicate a non-parabolic trend i.e., velocity profiles are blunted or flat.

## VI. CONCLUSION

In this paper a two-layered tube flow model of blood with plasma in the peripheral region and core a Herschel-Bulkley fluid subject to a velocity slip condition at interface, suggested by Chaturani and Biswas, Biswas and Mazumdar and Biswas and Nath and usual no-slip at tube has been investigated. Analytical expressions are obtained for velocity, flow rate, mean velocity and apparent viscosity. It can be noticed that velocity is a function of pressure gradient  $C$ , fluid viscosity  $k$ , plasma viscosity  $\mu_1$ , radial co-ordinate  $r$ , tube radius  $R$ , interface co-ordinate  $a$ , yield stress  $\tau_H$ , fluid property  $n$ , and axial velocity slip  $u_s$  at interface of fluids. For blood flow in narrow tubes, fluid index  $n$  plays an important role and flow variables are influenced by the variation of  $n$  ( $n <, =, > 1$ ). Apparent viscosity increases and decreases with tube radius and so it exhibits both Fahraeus-Linqvist Effect (ELF) and its inversion (IFLE). Theoretical analysis developed here, could be exploited for the fundamental understanding and development of many CVS and arterial diseases.

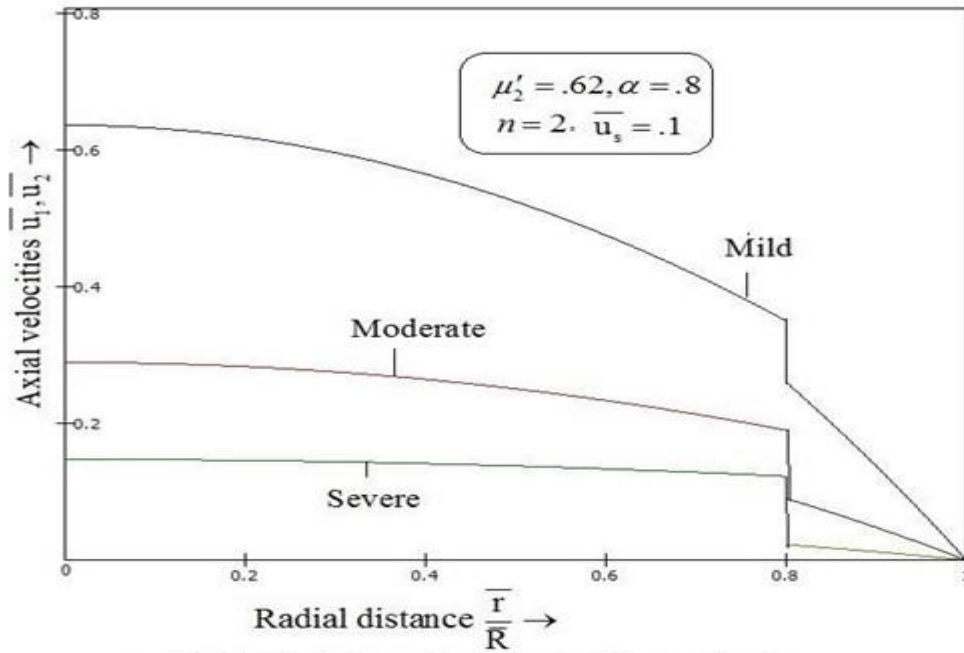


Fig. 2: Variation of axial velocities against radial distance and slip velocity  $\bar{u}_s = .1$

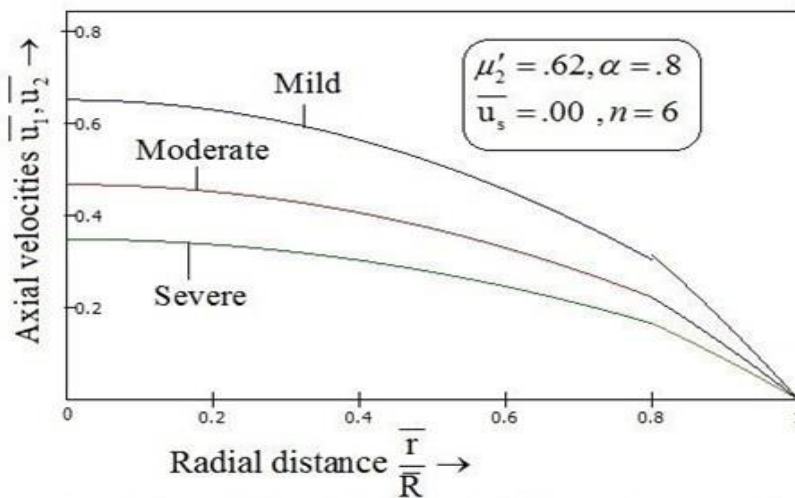


Fig. 5: Variation of axial velocities against radial distance for mild, moderate, severe,  $n=6$  and slip velocity  $\bar{u}_s = .00$



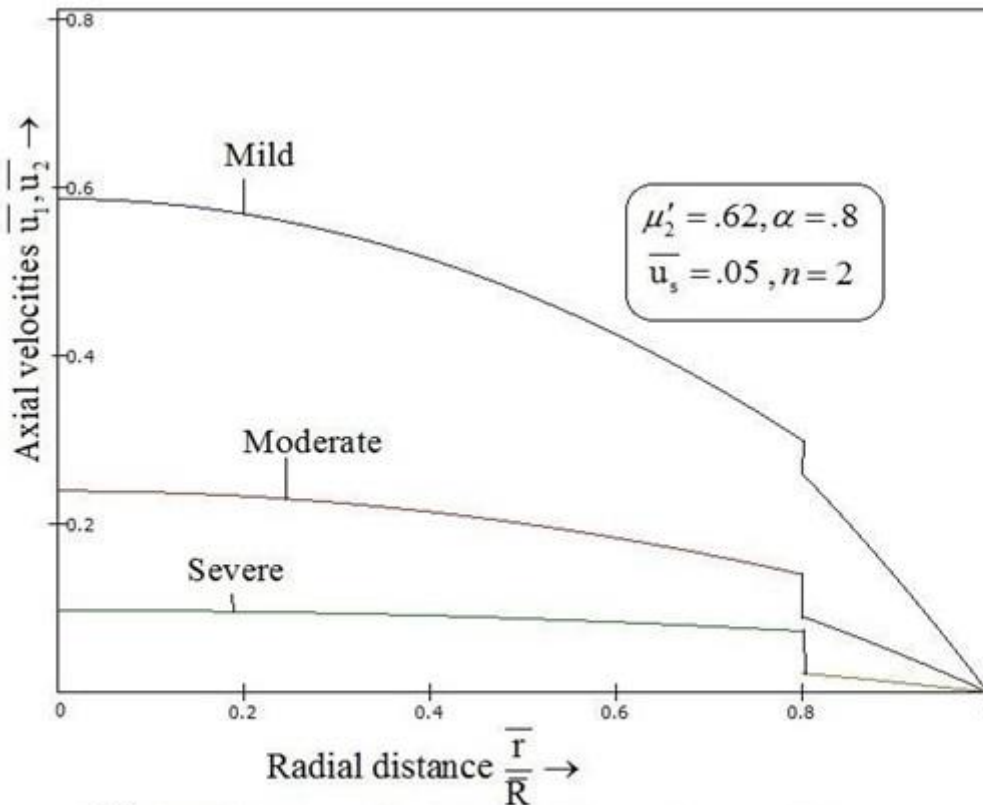


Fig.4 : Variation of axial velocities against radial distance for mild, moderate, severe,  $n=2$  and slip velocity  $\bar{u}_s = .05$

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