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Some Refinements of Certain Linear Discrete Inequalities applicable in Difference Equations

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ABSTRACT: In this manuscript, we state and prove some generalized and extended nonlinear estimates on certain existing discrete inequalities. These inequalities can find applications in the analysis of some discrete event dynamical systems to study boundedness, uniqueness, stability of the solutions of difference equations therein. We also provide some applications to convey the importance of our results.

KEYWORDS: Discrete inequalities, Difference equations, Boundedness, etc.

I. INTRODUCTION

Inequalities occupy a central position in the research domain, which concerns the study of several qualitative and quantitative properties of the solutions of differential, integral, fractional differential, fractional difference, and difference equations (see. [1], [2], [3], [4], [5]). It is a well-acquainted fact that the inequality technique is popularly employed to deal with the certain crucial analysis of discrete event dynamical systems. In numerous circumstances viz. Mathematical model analysis, biological processes, etc., one finds the mathematical equations, which involve cumbersome calculations in the search of exact solutions. In such cases, inequalities play a crucial role, since they provide explicit bounds on the solutions of the concerned equations. These bounds can be readily used to study distinct characteristics of the solutions like stability, existence, uniqueness, and many more.

In the year 1969, Sugiyama [2] presented the most complete discrete analogue of celebrated Gronwall-Bellman inequality, which has a wide range of applications in difference equations. Further, in 1973, Pachpatte[4] established a generalized version of Sugiyama's inequality, which can be adopted to solve certain crucial finite difference equations, where, Sugiyama's inequality fails to provide the explicit bounds. On the parallel lines, Pachpatte[1] [5], Gamidov[6], Ma [7], Zhao [8] stated and proved several further generalizations of the inequalities developed by Sugiyama [2] and Pachpatte[4].

In all over discussion, we take $N_0 = \{0, 1, 2, \dots\}$ and for any function $x(t), t \in N_0$, we define the operator Δ as $\Delta x(t) = x(t+1) - x(t)$. For $t_1 > t_2$, $t_1, t_2 \in N_0$ and any function $x(t), t \in N_0$, we use the usual notations: $\sum_{s=t_1}^{t_2} x(s) = 0$ and $\prod_{s=t_1}^{t_2} x(s) = 1$.

We list here some important lemmas and finite difference inequalities, before proceeding towards our main results.

Theorem 1.1. (Pachpatte's Inequality [1])

Let $u(t), a(t)$ and $b(t)$ be nonnegative functions defined on N_0 and

$$\Delta u(t) \leq a(t)u(t) + b(t), t \in N_0,$$

then

$$u(t) \leq u(0) \prod_{s=0}^{t-1} (1 + a(s)) + \sum_{s=0}^{t-1} b(s) \prod_{\sigma=s+1}^{t-1} (1 + a(\sigma)), t \in N_0.$$

Theorem 1.2. (Sugiyama's Inequality [2])

Let $u(t)$, $a(t)$, $b(t)$ be nonnegative functions defined on N_0 and

$$u(t) \leq a(t) + \sum_{s=0}^{t-1} b(s)u(s), t \in N_0,$$

Where $c \geq 0$ is a constant then

$$u(t) \leq a(t) + \sum_{s=0}^{t-1} a(s)b(s) \prod_{\sigma=s+1}^{t-1} (1 + b(\sigma)), t \in N_0.$$

Theorem 1.3. (Sugiyama's Inequality [2])

Let $u(t)$, $b(t)$ be nonnegative functions defined on N_0 and

$$u(t) \leq c + \sum_{s=0}^{t-1} b(s)u(s), t \in N_0,$$

Where $c \geq 0$ is a constant then

$$u(t) \leq c \prod_{s=0}^{t-1} (1 + b(s)), t \in N_0.$$

Lemma 1.4. (Zhao [8])

Assume that $a \geq 0, p \geq q \geq 0, p \neq 0$ then

$$a^{\frac{q}{p}} \leq \frac{q}{p} k^{\frac{q-p}{p}} a + \frac{p-q}{p} k^{\frac{q}{p}},$$

for any $k > 0$.

II. MAIN RESULTS

Theorem 2.1. Assume that $x(t), \alpha(t), \beta(t), f(t), g(t)$ be non-negative functions defined on N_0 . If, for $t \in N_0, x(t)$ satisfies

$$x^p(t) \leq \alpha(t) + f(t) \sum_{s=0}^{t-1} [\beta(s)x^q(s) + g(s)] \quad (1)$$

then

$$x(t) \leq \left(\alpha(t) + f(t) \left[\sum_{s=0}^{t-1} \left[\frac{q}{p} k^{\frac{q-p}{p}} \beta(s) \alpha(s) + \frac{p-q}{p} k^{\frac{q}{p}} \beta(s) + g(s) \right] \right. \right. \\ \left. \left. \times \prod_{\sigma=s+1}^{t-1} \left\{ 1 + \left[\frac{q}{p} k^{\frac{q-p}{p}} \beta(\sigma) f(\sigma) \right] \right\} \right] \right)^{\frac{1}{p}} \quad (2)$$

for $t \in N_0$.

Proof. Let us define a function $y(t)$ as

$$y(t) = \sum_{s=0}^{t-1} [\beta(s)x^q(s) + g(s)] \quad (3)$$

It gives $y(0) = 0$ and

$$x^p(t) \leq \alpha(t) + f(t)y(t) \quad (4)$$

Further from (3), (4), followed by using Lemma 1.4, we get

$$\begin{aligned} \Delta y(t) &= \beta(t)x^q(t) + g(t) \\ &\leq \beta(t)(\alpha(t) + f(t)y(t))^{\frac{q}{p}} + g(t) \\ &\leq \beta(t) \left[\frac{q}{p} k^{\frac{q-p}{p}} (\alpha(t) + f(t)y(t)) + \frac{p-q}{p} k^{\frac{q}{p}} \right] + g(t) \end{aligned}$$

Now making use of Theorem 1.1, we obtain

$$y(t) \leq \sum_{s=0}^{t-1} \left[\frac{q}{p} k^{\frac{q-p}{p}} \beta(s) \alpha(s) + \frac{p-q}{p} k^{\frac{q}{p}} \beta(s) + g(s) \right] \prod_{\sigma=s+1}^{t-1} \left\{ 1 + \left[\frac{q}{p} k^{\frac{q-p}{p}} \beta(\sigma) f(\sigma) \right] \right\}$$

Using this estimate in (4), we get the desired bound on $x(t)$ as stated in (2).

Remark.

1. If we substitute, $p = q = 1, f(t) = 1, g(t) = 0$ for each $t \in N_0$ then Sugiyama's inequality stated in Theorem 1.2 becomes a special case of Theorem 2.1.
2. Pachapatt's inequality ([1], p.19, Theorem 1.3.1) can be retraced as a particular case of Theorem 2.1 under the substitutions $p = q = 1$.

Theorem 2.2. Assume that $x(t), \alpha(t), f_k(t), \beta_k(t), k = 1, 2, 3, \dots, n$, are nonnegative functions defined on N_0 . If, for $t \in N_0, x(t)$ satisfies

$$x^p(t) \leq \alpha(t) + \sum_{k=1}^n f_k(t) \left(\sum_{s=0}^{t-1} \beta_k(s) x^q(s) \right) \quad (5)$$

then for $t \in N_0$

$$\begin{aligned} x(t) &\leq \left(\alpha(t) + f(t) \left[\sum_{s=0}^{t-1} \left[\frac{q}{p} k^{\frac{q-p}{p}} \left(\sum_{k=1}^n \beta_k(s) \right) \alpha(s) + \frac{p-q}{p} k^{\frac{q}{p}} \left(\sum_{k=1}^n \beta_k(s) \right) \right] \right. \right. \\ &\quad \left. \left. \times \prod_{\sigma=s+1}^{t-1} \left\{ 1 + \left[\frac{q}{p} k^{\frac{q-p}{p}} \left(\sum_{k=1}^n \beta_k(\sigma) \right) f(\sigma) \right] \right\} \right]^{\frac{1}{p}} \right) \end{aligned} \quad (6)$$

Proof. Let

$$f(t) = \sup_k f_k(t), t \in N_0$$

Then from (2.7), it is easy to observe that

$$x^p(t) \leq \alpha(t) + f(t) \sum_{s=0}^{t-1} \left(\sum_{k=1}^n \beta_k(s) \right) x^q(s).$$

An easy application of Theorem 2.1 implies the estimate

$$\begin{aligned} x(t) &\leq \left(\alpha(t) + f(t) \left[\sum_{s=0}^{t-1} \left[\frac{q}{p} k^{\frac{q-p}{p}} \left(\sum_{k=1}^n \beta_k(s) \right) \alpha(s) + \frac{p-q}{p} k^{\frac{q}{p}} \left(\sum_{k=1}^n \beta_k(s) \right) \right] \right. \right. \\ &\quad \left. \left. \times \prod_{\sigma=s+1}^{t-1} \left\{ 1 + \left[\frac{q}{p} k^{\frac{q-p}{p}} \left(\sum_{k=1}^n \beta_k(\sigma) \right) f(\sigma) \right] \right\} \right]^{\frac{1}{p}} \right) \end{aligned}$$

which is the desired bound as stated in (6).

Remark. Gamidov's inequality ([1], p.19, Theorem 1.3.2) can be achieved as a special version of Theorem 2.2, if we consider, $p = q = 1$.

Further, we present certain inequalities to deal with the more general class of finite difference equations, which involve the function of two independent variables like, $\Delta x^p(t) = \partial(t, s)x^q(t)$, where $x(0) = c^{\frac{1}{p}}$, for some $c \geq 0, p \geq q \geq 0, p \neq 0$. We begin with the following result:

Theorem 2.3. Assume that $x(t)$ is nonnegative function defined on N_0 and let $J(t, s), 0 \leq s \leq t$ be nonnegative and nondecreasing function in t for every $s \in N_0$. If for $c \geq 0$

$$x^p(t) \leq c + \sum_{s=0}^{t-1} J(t, s)x^q(s), \text{ where } 0 \leq s \leq t, s, t \in N_0, p \geq q \geq 0, p \neq 0$$

then

$$x(t) \leq \left\{ c \prod_{s=0}^{t-1} \left(1 + \frac{q}{p} k^{\frac{a-p}{p}} \partial(t, s) \right) + \sum_{s=0}^{t-1} \frac{p-q}{p} k^{\frac{a}{p}} J(t, s) \prod_{\sigma=s+1}^{t-1} \left(1 + \frac{q}{p} k^{\frac{q-p}{p}} \partial(t, \sigma) \right) \right\}^{\frac{1}{p}}$$

Proof. At the first, fix any $T \in N_0$, then

$$x^p(t) \leq c + \sum_{s=0}^{t-1} J(T, s)x^q(s), \text{ for } 0 \leq t \leq T$$

Let us define a function $y(t)$ as

$$y(t) = c + \sum_{s=0}^{t-1} J(T, s)x^q(s)$$

This gives $y(0) = c$ and

$$x^p(t) \leq y(t), \text{ for } 0 \leq t \leq T \quad (7)$$

Further

$$\begin{aligned} \Delta y(t) &= \partial(T, t)x^q(t) \\ &\leq \partial(T, t)y^{\frac{a}{p}}(t) \\ &\leq J(T, t) \left\{ \frac{q}{p} k^{\frac{a-p}{p}} y(t) + \frac{p-q}{p} k^{\frac{a}{p}} \right\} \\ \Delta y(t) &\leq J(T, t) \frac{q}{p} k^{\frac{q-p}{p}} y(t) + \frac{p-q}{p} k^{\frac{a}{p}} \partial(T, t) \end{aligned}$$

An application of Theorem 1.1 implies the bound

$$y(t) \leq c \prod_{s=0}^{t-1} \left(1 + \frac{q}{p} k^{\frac{a-p}{p}} \partial(T, s) \right) + \sum_{s=0}^{t-1} \frac{p-q}{p} k^{\frac{a}{p}} \partial(T, s) \prod_{\sigma=s+1}^{t-1} \left(1 + \frac{q}{p} k^{\frac{q-p}{p}} \partial(T, \sigma) \right) \quad (8)$$

However, since T is arbitrary, (8) together with (7) gives

$$x(t) \leq \left\{ c \prod_{s=0}^{t-1} \left(1 + \frac{q}{p} k^{\frac{q-p}{p}} \partial(t, s) \right) + \sum_{s=0}^{t-1} \frac{p-q}{p} k^{\frac{a}{p}} \partial(t, s) \prod_{\sigma=s+1}^{t-1} \left(1 + \frac{q}{p} k^{\frac{q-p}{p}} \partial(t, \sigma) \right) \right\}^{\frac{1}{p}}$$

for $t \in N_0, k > 0$.

Remark.

1. Sugiyama's inequality given in Theorem 1.3 is a specific case of inequality proved in Theorem 2.3, if we let, $p = q = 1$ and $J(t, s) = b(s), t \in N_0$.
2. Pachapatt's inequality ([1], p.20, Theorem 1.3.3) is a particular case of Theorem 2.1 under the consideration $p = q = 1$.

Theorem 2.4. Assume that $x(t)$ is nonnegative function defined on N_0 and let $\partial(t, s), 0 \leq s \leq t$ be nonnegative and nondecreasing function in t for every $s \in N_0$. If for nonnegative, and nondecreasing function $\alpha(t), t \in N_0$,

$$x^p(t) \leq \alpha(t) + \sum_{s=0}^{t-1} J(t, s)x(s), \text{ where } 0 \leq s \leq t, s, t \in N_0, p \geq 1$$

then

$$x(t) \leq \alpha(t) \left\{ \prod_{s=0}^{t-1} \left(1 + \frac{1}{p} k^{\frac{1-p}{F}} \partial(t, s) \right) + \sum_{s=0}^{t-1} \frac{p-1}{p} k^{\frac{1}{p}} \partial(t, s) \prod_{\sigma=s+1}^{t-1} \left(1 + \frac{1}{p} k^{\frac{1-p}{F}} \partial(t, \sigma) \right) \right\}^{\frac{1}{p}}$$

for $t \in N_0, k > 0$.

Proof. We begin with the assumption that $\alpha(t) > 0, t \in N_0$. Since $p \geq 1$ and $\alpha(t) > 0$, we obtain

$$\frac{x^p(t)}{\alpha^p(t)} \leq 1 + \sum_{s=0}^{t-1} J(t, s) \frac{x(s)}{\alpha(s)} \quad (9)$$

Substitute

$$\frac{x(t)}{\alpha(t)} = u(t)$$

then (9) reduces to

$$u^p(t) \leq 1 + \sum_{s=0}^{t-1} J(t, s)u(s)$$

The straightforward application of Theorem 2.3 with $q = 1, c = 1$ implies the bound

$$u(t) \leq \left\{ \prod_{s=0}^{t-1} \left(1 + \frac{1}{p} k^{\frac{1-p}{p}} \partial(t, s) \right) + \sum_{s=0}^{t-1} \frac{p-1}{p} k^{\frac{1}{2}} g(t, s) \prod_{\sigma=s+1}^{t-1} \left(1 + \frac{1}{p} k^{\frac{1-p}{p}} \partial(t, \sigma) \right) \right\}^{\frac{1}{p}}$$

for $t \in N_0, k > 0$, which together with (9) gives that

$$x(t) \leq \alpha(t) \left\{ \prod_{s=0}^{t-1} \left(1 + \frac{1}{p} k^{\frac{1-p}{p}} \partial(t, s) \right) + \sum_{s=0}^{t-1} \frac{p-1}{p} k^{\frac{1}{p}} \partial(t, s) \prod_{\sigma=s+1}^{t-1} \left(1 + \frac{1}{p} k^{\frac{1-p}{F}} \partial(t, \sigma) \right) \right\}^{\frac{1}{p}}$$

for $t \in N_0, k > 0$.

Theorem 2.5. Assume that $x(t), \alpha(t), \beta(t), f(t)$ and $g(t)$ are nonnegative functions defined on N_0 . Let $\partial(t, s), \Delta_t J(t, s) = J(t+1, s) - J(t, s)$ be nonnegative functions for $0 \leq s \leq t, s, t \in N_0$. If, for $p \geq q \geq 0, p \neq 0, x(t)$ satisfies

$$x^p(t) \leq \alpha(t) + f(t) \sum_{s=0}^{t-1} J(t, s)(\beta(s)x^q(s) + g(s)), t \in N_0$$

then

$$x(t) \leq \alpha(t) + f(t) \sum_{s=0}^{t-1} \mathcal{B}(s) \prod_{\sigma=s+1}^{t-1} (1 + \mathcal{A}(\sigma)), t \in N_0$$

where.

$$\begin{aligned} \mathcal{A}(t) &= J(t+1, t) \frac{q}{p} k^{\frac{a-p}{p}} \beta(t) f(t) + \sum_{s=0}^{t-1} \Delta_t J(t, s) \frac{q}{p} k^{\frac{a-p}{p}} \beta(s) f(s) \\ \mathcal{B}(t) &= \partial(t+1, t) \left[\frac{q}{p} k^{\frac{q-p}{p}} \beta(t) \alpha(t) + \frac{p-q}{p} k^{\frac{q}{p}} \beta(t) + g(t) \right] \\ &\quad + \sum_{s=0}^{t-1} \Delta_t \partial(t, s) \left[\frac{q}{p} k^{\frac{q-p}{p}} \beta(s) \alpha(s) + \frac{p-q}{p} k^{\frac{q}{p}} \beta(s) + g(s) \right] \end{aligned}$$

$0 \leq s \leq t, s, t \in N_0$.

Proof. We begin with the function $y(t)$ defined as

$$y(t) = \sum_{s=0}^{t-1} J(t, s)(\beta(s)x^q(s) + g(s)), t \in N_0$$

This gives $y(0) = 0$ and

$$x^p(t) \leq \alpha(t) + f(t)y(t), t \in N_0 \quad (10)$$

It is simple to notice that $y(t)$ is monotonically nondecreasing in t , thus

$$\begin{aligned} \Delta y(t) &= \sum_{s=0}^t J(t+1, s)(\beta(s)x^q(s) + g(s)) - \sum_{s=0}^{t-1} \partial(t, s)(\beta(s)x^q(s) + g(s)) \\ &= J(t+1, t)(\beta(t)x^q(t) + g(t)) + \sum_{s=0}^{t-1} [\partial(t+1, s) - \partial(t, s)](\beta(s)x^q(s) + g(s)) \\ &= J(t+1, t)(\beta(t)x^q(t) + g(t)) + \sum_{s=0}^{t-1} \Delta_t J(t, s)(\beta(s)x^q(s) + g(s)) \\ &\leq \partial(t+1, t) \left(\beta(t) \{ \alpha(t) + f(t)y(t) \}^{\frac{q}{p}} + g(t) \right) \\ &\quad + \sum_{s=0}^{t-1} \Delta_t \partial(t, s) \left(\beta(s) \{ \alpha(s) + f(s)y(s) \}^{\frac{q}{p}} + g(s) \right) \\ &\leq \partial(t+1, t) \left(\beta(t) \left\{ \frac{q}{p} k^{\frac{q-p}{p}} [\alpha(t) + f(t)y(t)] + \frac{p-q}{p} k^{\frac{q}{p}} \right\} + g(t) \right) \\ &\quad + \sum_{s=0}^{t-1} \Delta_t \partial(t, s) \left(\beta(s) \left\{ \frac{q}{p} k^{\frac{q-p}{p}} [\alpha(s) + f(s)y(s)] + \frac{p-q}{p} k^{\frac{q}{p}} \right\} + g(s) \right) \\ &= \left\{ \partial(t+1, t) \frac{q}{p} k^{\frac{q-p}{p}} \beta(t) f(t) + \sum_{s=0}^{t-1} \Delta_t \partial(t, s) \frac{q}{p} k^{\frac{q-p}{p}} \beta(s) f(s) \right\} y(t) \\ &\quad + \left\{ \partial(t+1, t) \left[\frac{q}{p} k^{\frac{q-p}{p}} \beta(t) \alpha(t) + \frac{p-q}{p} k^{\frac{q}{p}} \beta(t) + g(t) \right] \right. \\ &\quad \left. + \sum_{s=0}^{t-1} \Delta_t J(t, s) \left[\frac{q}{p} k^{\frac{q-p}{p}} \beta(s) \alpha(s) + \frac{p-q}{p} k^{\frac{q}{p}} \beta(s) + g(s) \right] \right\} \\ &= \mathcal{A}(t)y(t) + \mathcal{B}(t) \end{aligned}$$

An effortless use of Theorem 1.1 provides the bound

$$y(t) \leq \sum_{s=0}^{t-1} \mathcal{B}(s) \prod_{\sigma=s+1}^{t-1} (1 + \mathcal{A}(\sigma)) \quad (11)$$

Further, combined use of (10) and (11) implies that

$$x(t) \leq \alpha(t) + f(t) \sum_{s=0}^{t-1} \mathcal{B}(s) \prod_{\sigma=s+1}^{t-1} (1 + \mathcal{A}(\sigma)), t \in N_0$$

This completes the proof of our Theorem.

Remark. We can acquire an handy and fascinating inequality established by Pachappte ([1], p.13, Theorem 1.2.3) as a definite case of Theorem 2.5, if we put $p = q = 1$.

III. APPLICATIONS

Example 3.1. Consider the difference equation

$$x^3(t) = 2t^2 + \sum_{s=0}^{t-1} x^2(s) + s + 1, t \in N_0 \quad (12)$$

Let us assume that the functions and indices involved in (3.1) are as follows:

$$\alpha(t) = 2t^2, \quad \beta(t) = 1 = f(t), \quad g(t) = t + 1, \quad p = 3, \quad \text{and} \quad q = 2$$

Thus, an application of Theorem 2.1, we obtain

$$x(t) \leq \left(2t^2 + \left[\sum_{s=0}^{t-1} \left[\frac{2}{3} k^{\frac{-1}{3}} 2s^2 + \frac{1}{3} k^{\frac{2}{3}} + s + 1 \right] \times \prod_{\sigma=s+1}^{t-1} \left\{ 1 + \left[\frac{2}{3} k^{\frac{-1}{3}} \right] \right\} \right] \right)^{\frac{1}{3}}$$

On computing the right-hand side of inequality (3.2), we get

$$\begin{aligned} x(t) &\leq \left(2t^2 + \frac{1}{4} \left(45k^{\frac{2}{3}} \left(\left(\frac{2}{3\sqrt[3]{k}} + 1 \right)^t - 1 \right) + 6\sqrt[3]{k} \left(3 \left(\frac{2}{3\sqrt[3]{k}} + 1 \right)^t - 5t - 3 \right) \right. \right. \\ &\quad \left. \left. + 2k \left(\left(\frac{2}{3\sqrt[3]{k}} + 1 \right)^t - 1 \right) - 8t^2 \right) \right)^{\frac{1}{3}} \\ &\leq \left(\frac{1}{4} \left(45k^{\frac{2}{3}} \left(\left(\frac{2}{3\sqrt[3]{k}} + 1 \right)^t - 1 \right) + 6\sqrt[3]{k} \left(3 \left(\frac{2}{3\sqrt[3]{k}} + 1 \right)^t - 5t - 3 \right) \right. \right. \\ &\quad \left. \left. + 2k \left(\left(\frac{2}{3\sqrt[3]{k}} + 1 \right)^t - 1 \right) \right) \right)^{\frac{1}{3}} \\ x(t) &\leq \frac{\sqrt[3]{45k^{\frac{2}{3}} \left(\left(\frac{2}{3\sqrt[3]{k}} + 1 \right)^t - 1 \right) + 6\sqrt[3]{k} \left(3 \left(\frac{2}{3\sqrt[3]{k}} + 1 \right)^t - 5t - 3 \right) + 2k \left(\left(\frac{2}{3\sqrt[3]{k}} + 1 \right)^t - 1 \right)}}{2^{2/3}} \end{aligned} \quad (13)$$

For instance, if we take $k = 1$, then the bound on the solution of equation (12), using (13) is found to be

$$x(t) \leq \frac{\sqrt[3]{47 \left(\left(\frac{5}{3} \right)^t - 1 \right) + 6(-5t + 3^{1-t}5^t - 3)}}{2^{2/3}}$$

Example 3.2. Consider the difference equation

$$x^2(t) = 4 + \sum_{s=0}^{t-1} (t^2 + s)x(s) \quad (14)$$

If we let $\partial(t, s) = t^2 + s$ and $c = 4$, then an application of Theorem 2.3 provides the estimate on the bound of solution as

$$x(t) \leq \left\{ 4 \prod_{s=0}^{t-1} \left(1 + \frac{1}{2} k^{\frac{-1}{2}} (t^2 + s) \right) + \sum_{s=0}^{t-1} \frac{1}{2} k^{\frac{1}{2}} (t^2 + s) \prod_{\sigma=s+1}^{t-1} \left(1 + \frac{1}{2} k^{\frac{-1}{2}} (t^2 + \sigma) \right) \right\}^{\frac{1}{2}}$$

On computing the right-hand side of inequality (3.6), we get

$$x(t) \leq \left\{ \frac{(k+4)2^{-t}k^{-\frac{t}{2}}\Gamma(t^2+t+2\sqrt{k})}{\Gamma(t^2+2\sqrt{k})} - k \right\}^{\frac{1}{2}} \quad (15)$$

For instance, if we let $k = 1$, then the estimate on bound of the solution of equation (14), using (15) is

$$x(t) \leq \left\{ \frac{52^{-t}\Gamma(t^2+t+2)}{\Gamma(t^2+2)} - 1 \right\}^{\frac{1}{2}}$$

IV. CONCLUSIONS

In this article, we have obtained certain useful generalizations of some prominent discrete inequalities due to Pachpatte, Gamidov, and Sugiyama. These results can play a crucial role in the determination of an explicit bound on the solutions of a class of nonlinear difference equations. However, we have to deal with several more general and



complicated difference equations in the modeling of numerous biological, ecological processes. Thus, these results can be further generalized to undertake the study of several general classes of difference equations.

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