



e-ISSN: 2319-8753 | p-ISSN: 2320-6710

IJIRSET

International Journal of Innovative Research in
SCIENCE | ENGINEERING | TECHNOLOGY



INTERNATIONAL JOURNAL OF INNOVATIVE RESEARCH

IN SCIENCE | ENGINEERING | TECHNOLOGY

Volume 11, Issue 5, May 2022

ISSN INTERNATIONAL
STANDARD
SERIAL
NUMBER
INDIA

Impact Factor: 8.118



9940 572 462



6381 907 438



ijirset@gmail.com



www.ijirset.com

Lattice Points for the Quadratic Diophantine Equation

$$21(x^2 + y^2) - 19xy = 84z^2$$

Manju Somanath¹, T.Anupreethi², V.Sangeetha³

Assistant Professor, Department of Mathematics, National College, Trichy, Tamil Nadu, India.^{1,3}

Research Scholar, Department of Mathematics, National College, Trichy, Tamil Nadu, India.²

ABSTRACT: The ternary quadratic diophantine equation $21(x^2 + y^2) - 19xy = 84z^2$ is analysed for its non-zero integral solutions. Different patterns of solutions are obtained by employing the methods of factorization, cross multiplication and the concept of double equations.

KEYWORDS: Diophantine equation, integral solutions, lattice points.

I. INTRODUCTION

The Diophantine equations are equations involving only sums, products and powers in which all the constants are integers and the only solutions of interest are integers. For example,

$3x + 7y = 1$ or $x^2 - y^2 = z^2$ where x, y and z are integers. Named in honour of the 3rd century Greek Mathematician Diophantus of Alexandria, these equations were first systematically solved by Indian Mathematicians beginning with Aryabhata. Diophantine equations fall into three classes: those with no solutions, those with only finitely many solutions and those with infinitely many solutions. Many methods are available to solve these types of equations but there is no universal method to determine whether the solutions to diophantine equation exists or not.

In this paper, an interesting quadratic diophantine equation with three unknowns

$21(x^2 + y^2) - 19xy = 84z^2$ is discussed and solved for its non-zero integral solutions. Different patterns of solutions are obtained by employing various methods.

II. METHOD OF ANALYSIS

The ternary quadratic diophantine equation to be solved for its non-zero distinct integral solution is $21(x^2 + y^2) - 19xy = 84z^2$(1). The introduction of the linear transformations

$x = u + v, y = u - v, u \neq 0, v \neq 0$ (2) reduces (1) to $23u^2 + 61v^2 = 84z^2$ (3). Different patterns of solutions of (1) are presented below.

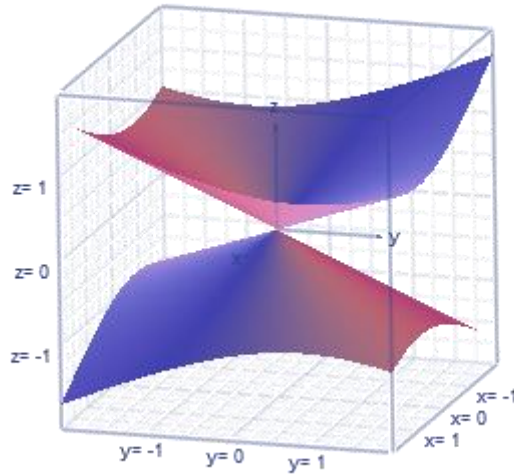


Figure 1: Pictorial representation of the equation

Choice I: The complex number factorization of 84 is employed to obtain two patterns of solutions.

Pattern-1:

Consider 84 as $84 = (\sqrt{23} + i\sqrt{61})(\sqrt{23} - i\sqrt{61}) \dots \dots (4)$ and assuming the value of z as

$$z = 23a^2 + 61b^2 \dots \dots (5) \text{ for the non-zero integers } a \text{ and } b.$$

The substitution of (4) and (5) in (3) yields

$$\sqrt{23}u + i\sqrt{61}v = (\sqrt{23} + i\sqrt{61})(\sqrt{23}a + i\sqrt{61}b)^2 \dots \dots (6)$$

On equating the real and imaginary parts of (6) we get

$$u = 23a^2 - 61b^2 - 122ab \text{ and } v = 23a^2 - 61b^2 + 46ab$$

From these values, we can find the non-zero integral solutions of (1) as

$$x = 46a^2 - 122b^2 - 76ab; y = -168ab; z = 23a^2 + 61b^2.$$

Pattern-2:

The choice of 84 as $84 = (-\sqrt{23} + i\sqrt{61})(-\sqrt{23} - i\sqrt{61})$ and $z = 23a^2 + 61b^2$

in (3) and following the method as above, we get the integral solutions of (1) as

$$x = -168ab; y = -46a^2 + 122b^2 - 76ab; z = 23a^2 + 61b^2.$$

Choice II: Two different patterns of solutions are derived from the irrational number factorization of 61 and 23.

Pattern-3:

Consider 61 as $61 = (\sqrt{84} + \sqrt{23})(\sqrt{84} - \sqrt{23}) \dots \dots (7)$ and assuming the value of v as



$$v = 84a^2 - 23b^2 \dots\dots\dots (8) \text{ for the non-zero integers } a \text{ and } b.$$

The substitution of (7) and (8) in (3) yields

$$\sqrt{84}z + \sqrt{23}u = (\sqrt{84} + \sqrt{23})(\sqrt{84}a + \sqrt{23}b)^2 \dots\dots\dots (9)$$

On equating the irrational parts of (9), we get

$$z = 84a^2 + 23b^2 + 46ab \text{ and } u = 84a^2 + 23b^2 + 168ab$$

From these values, we can find the non-zero integral solutions of (1) as

$$x = 168a^2 + 168ab; y = 46b^2 + 168ab; z = 84a^2 + 23b^2 + 46ab.$$

Pattern-4:

The choice of 23 as $23 = (\sqrt{84} + \sqrt{61})(\sqrt{84} - \sqrt{61})$ and assuming $u = 84a^2 - 61b^2$ in (3) and following the method as above, we get the integral solutions of (1) as $x = 168a^2 + 168ab; y = -122b^2 - 168ab; z = 84a^2 + 61b^2 + 122ab$.

Choice III: Employing the method of cross multiplication, we have computed the integral solutions of (1) in 6 different patterns.

Pattern-5:

Consider (3) as $23(u^2 - z^2) = 61(z^2 - v^2) \dots\dots\dots(10)$ which can be written in the ratio form as

$$\frac{23(u+z)}{61(z+v)} = \frac{(z-v)}{(u-z)} = \frac{\alpha}{\beta}, \beta \neq 0. \text{ which is equivalent to the following two equations}$$

$$-\alpha u - \beta v + z(\alpha + \beta) = 0; 23u\beta - 61\alpha v + z(23\beta - 61\alpha) = 0$$

Solving the above equations, we get

$$u = 61\alpha^2 - 23\beta^2 + 122\alpha\beta; v = -61\alpha^2 + 23\beta^2 + 46\alpha\beta; z = 61\alpha^2 + 23\beta^2$$

From these values, we can find non-zero integral solutions to (1) as

$$x = 168\alpha\beta; y = 122\alpha^2 - 46\beta^2 + 76\alpha\beta; z = 61\alpha^2 + 23\beta^2.$$

Following the above process, we get the different patterns of integral solutions to (1) and are presented below.

Table: 1 System of equations and its solutions

Pattern	System Of Equations	Solutions
6	$23\beta u - \alpha v + z(23\beta - \alpha) = 0$ $-\alpha u - 61\beta v + z(61\beta + \alpha) = 0$	$x = -168\alpha\beta$ $y = -2\alpha^2 + 2806\beta^2 - 76\alpha\beta$ $z = -\alpha^2 - 1403\beta^2$

7	$23u\beta - \alpha v - z(23\beta + \alpha) = 0$ $-\alpha u - 61\beta v + z(61\beta - \alpha) = 0$	$x = 2\alpha^2 - 2806\beta^2 - 76\alpha\beta$ $y = -168\alpha\beta$ $z = -\alpha^2 - 1403\beta^2$
8	$23\beta u - 61\alpha v - z(23\beta + 61\alpha) = 0$ $-\alpha u - v\beta + z(\beta - \alpha) = 0$	$x = 122\alpha^2 - 46\beta^2 - 76\alpha\beta$ $y = -168\alpha\beta$ $z = -61\alpha^2 - 23\beta^2$
9	$\beta u - \alpha v - z(\beta + \alpha) = 0$ $-23\alpha u - 61\beta v + z(61\beta - 23\alpha) = 0$	$x = 46\alpha^2 - 122\beta^2 - 76\alpha\beta$ $y = -168\alpha\beta$ $z = -23\alpha^2 - 61\beta^2$
10	$\beta u - 61\alpha v - z(61\alpha + \beta) = 0$ $-23\alpha u - v\beta + z(\beta - 23\alpha) = 0$	$x = 2806\alpha^2 - 2\beta^2 - 76\alpha\beta$ $y = -168\alpha\beta$ $z = -1403\alpha^2 - \beta^2$

Choice IV: Applying linear transformations and solving the corresponding equation in the concept of double equations, we have yet another patterns of integral solutions to (1).

Pattern-11:

Consider $z = X + 61T$, $v = X + 84T$ (11) reduces (3) to $X^2 = u^2 + 5124T^2$ (12)

which is satisfied by $T = 2mn$; $X = m^2 + 5124n^2$; $u = m^2 - 5124n^2$ (13)

The substitution of (13) in (11) yields, $z = m^2 + 5124n^2 + 122mn$; $u = m^2 - 5124n^2$;

$v = m^2 + 5124n^2 + 168mn$. From these values, we can find non-zero integral solutions to (1)

as $x = 2m^2 + 168mn$; $y = -10248n^2 - 168mn$; $z = m^2 + 5124n^2 + 122mn$.

Equation (12) can be expressed as a system of double equations in which the factorization employed is given in the following table.

Table: 2 Factorization for double equations

System	$X + u$	$X - u$
1	$7T^2$	732
2	732	$7T^2$
3	$732T$	$7T$
4	$7T$	$732T$
5	$3T^2$	1708
6	1708	$3T^2$
7	$1708T$	$3T$
8	$3T$	$1708T$

9	4	$1281T^2$
10	$1281T^2$	4
11	$1281T$	$4T$
12	$4T$	$1281T$
13	$21T^2$	244
14	244	$21T^2$
15	$244T$	$21T$
16	$21T$	$244T$
17	$427T^2$	12
18	$427T$	$12T$
19	$12T$	$427T$
20	$28T$	$183T$
21	$183T$	$28T$
22	28	$183T^2$
23	$61T^2$	84
24	$42T^2$	122
25	$122T$	$42T$
26	122	$42T^2$
27	42	$122T^2$
28	$5124T$	T

Solving each of the system of double equations and using (2) and (11), the corresponding integral solutions to (1) are found and they are presented below.

System:1 $x = 168k + 28k^2; y = -168k - 732; z = 366 + 122k + 14k^2$.

System:2 $x = 168k + 732; y = -168k - 28k^2; z = 366 + 122k + 14k^2$.

System:3 $x = 1632k; y = -182k; z = 861k$.

System:4 $x = 182k; y = -1632k; z = 861k$.

System:5 $x = 168k + 12k^2; y = -1708 - 168k; z = 854 + 122k + 6k^2$.

System:6 $x = 1708 + 168k; y = -168k - 12k^2; z = 854 + 122k + 6k^2$.

System:7 $x = 3584k; y = -174k; z = 1833k$.

System:8 $x = 174k; y = -3584k; z = 1833k$.

System:9 $x = 4 + 168k; y = -168k - 5124k^2; z = 2 + 122k + 2562k^2$.

System:10 $x = 5124k^2 + 168k; y = -4 - 168k; z = 2 + 122k + 2562k^2$.

System:11 $x = 5460k; y = -352k; z = 2814k$.

System:12 $x = 176k; y = -2730k; z = 1407k$.

System:13 $x = 168k + 84k^2; y = -244 - 168k; z = 122 + 122k + 42k^2$.

System:14 $x = 336k + 244; y = -336k - 336k^2; z = 122 + 244k + 168k^2$.



System:15 $x = 656k; y = -210k; z = 387k$.

System:16 $x = 420k; y = -1312k; z = 774k$.

System:17 $x = 336k + 6832k^2; y = -12 - 336k; z = 6 + 244k + 3416k^2$.

System:18 $x = 1022k; y = -192k; z = 561k$.

System:19 $x = 384k; y = -2044k; z = 1122k$.

System:20 $x = 224k; y = -534k; z = 333k$.

System:21 $x = 1068k; y = -448k; z = 666k$.

System:22 $x = 336k + 28; y = -336k - 2928k^2; z = 14 + 244k + 1464k^2$.

System:23 $x = 168k + 244k^2; y = -168k - 84; z = 42 + 122k + 122k^2$.

System:24 $x = 672k^2 + 336k; y = -122 - 336k; z = 366k^2 + 244k + 61$.

System:25 $x = 824k; y = -504k; z = 572k$.

System:26 $x = 168k + 122; y = -168k^2 - 168k; z = 61 + 122k + 84k^2$.

System:27 $x = 42 + 336k; y = -336k - 1952k^2; z = 21 + 244k + 976k^2$.

System:28 $x = 10416k; y = -170k; z = 5247k$.

Pattern-12:

Consider $z = X + 23T; u = X + 84T$,reduces (3) to $X^2 = v^2 + 1932T^2$(14)

and following the method as above, we get the integral solutions of (1) as

$$x = 2m^2 + 168mn; y = 3864n^2 + 168mn; z = m^2 + 1932n^2 + 46mn.$$

Equation (14) can be expressed as a system of double equations in which the factorization employed is given in the following table.

Table: 3 Factorization for double equations

System	$X + v$	$X - v$
1	$23T^2$	84
2	84	$23T^2$
3	$23T$	$84T$
4	$84T$	$23T$
5	$161T^2$	12
6	12	$161T^2$
7	$12T$	$161T$
8	$161T$	$12T$
9	92	$21T^2$
10	$21T$	$92T$
11	$92T$	$21T$



12	$69T^2$	28
13	28	$69T^2$
14	28T	69T
15	69T	28T
16	$483T^2$	4
17	4	$483T^2$
18	276T	7T
19	644T	3T
20	T^2	1932
21	1932T	T

Solving each of the system of double equations, the corresponding integral solutions to (1) are given below.

System:1 $x = 92k^2 + 168k; y = 84 + 168k; z = 46k^2 + 46k + 42$.

System:2 $x = 84 + 336k; y = 368k^2 + 336k; z = 184k^2 + 92k + 42$.

System:3 $x = 214k; y = 336k; z = 153k$.

System:4 $x = 672k; y = 428k; z = 306k$.

System:5 $x = 644k^2 + 168k; y = 12 + 168k; z = 322k^2 + 46k + 6$.

System:6 $x = 12 + 336k; y = 2576k^2 + 336k; z = 1288k^2 + 92k + 6$.

System:7 $x = 192k; y = 490k; z = 219k$.

System:8 $x = 980k; y = 384k; z = 438k$.

System:9 $x = 92 + 168k; y = 84k^2 + 168k; z = 42k^2 + 46k + 46$.

System:10 $x = 210k; y = 352k; z = 159k$.

System:11 $x = 704k; y = 420k; z = 318k$.

System:12 $x = 276k^2 + 168k; y = 28 + 168k; z = 138k^2 + 46k + 14$.

System:13 $x = 28 + 336k; y = 1104k^2 + 336k; z = 552k^2 + 92k + 14$.

System:14 $x = 224k; y = 306k; z = 143k$.

System:15 $x = 612k; y = 448k; z = 286k$.

System:16 $x = 1932k^2 + 168k; y = 4 + 168k; z = 966k^2 + 46k + 2$.

System:17 $x = 4 + 336k; y = 7728k^2 + 336k; z = 3864k^2 + 92k + 2$.

System:18 $x = 720k; y = 182k; z = 329k$.

System:19 $x = 1456k; y = 174k; z = 693k$.

System:20 $x = 16k^2 + 336k; y = 1932 + 336k; z = 966 + 92k + 8k^2$.



System: 21 $x = 4032k$; $y = 170k$; $z = 1979k$.

III. CONCLUSION

Quadratic diophantine equations in two, three or four unknowns is very important in the study of diophantine equations. Various methods are available to solve diophantine equations. This paper emphasizes infinite number of solutions to a particular diophantine equation in three unknowns.

REFERENCES

- [1] Andre Weil, Number Theory: An Approach through History, From Hammurapito to Legendre, Birkhäuser, Boston, 1987.
- [2] Bibhotibhusan Batta and Avadhesh Narayanan Singh, History of Hindu Mathematics, Asia Publishing House, 1983.
- [3] Boyer. C.B., History of Mathematics, John Wiley & sons Inc., New York, 1968.
- [4] L.E. Dickson, History of Theory of Numbers, Vol.2, Chelsea Publishing Company, New York, 1952.
- [5] L.J.Mordel, Diophantine Equations, Academic Press, New York, (1969).
- [6] Tituandrescu, Dorin Andrica, "An introduction to Diophantine equations" Springer Publishing House, 2002.
- [7] Manju Somanath, J. Kannan, K. Raja, "Integral solutions of an infinite elliptic cone", International Journal of Innovative Research in science, Engineering and Technology, 5(10) (2016), 17549-17557.
- [8] Manju Somanath, J. Kannan and K. Raja, "On Polynomial Solutions of Quadratic Equation", International Journal of Mathematics and its Applications (IJMAA), 5(4) (2017), 839-844.
- [9] Manju Somanath, J. Kannan and K. Raja, "On A Class of Solutions for a Diophantine Equation of Second Degree", International Journal of Pure and Applied Mathematics, 117 (12) (2017), 55-62.
- [10] J. Kannan, Manju Somanath and K. Raja, "Solutions of Negative Pell Equation Involving Twin Prime", JP Journal of Algebra, Number Theory and Applications, 40 (5) (2018), 869-874.
- [11] Manju Somanath, K. Raja, J. Kannan and M. Mahalakshmi, "On A Class Of Solutions For A Quadratic Diophantine Equation", Advances and Applications in Mathematical Sciences, 19 (11) (2020), 1097-1103.
- [12] M.A. Gopalan, J. Kannan, Manju Somanath and K. Raja, "Integral Solutions of an Infinite Elliptic Cone", International Journal of Innovative Research in Science, Engineering and Technology, (IJIRSET), 5(10) (2016), 17551-17557.
- [13] Manju Somanath, K. Raja, J. Kanna and B. Jeyashree, "Non Trivial Integral Solutions Of Ternary Quadratic Diophantine Equation", Advances and Applications in Mathematical Sciences, 19(11) (2020), 1105-1112.
- [14] J. Kannan, Manju Somanath and K. Raja, "On a Class of Solutions for the Hyperbolic Diophantine Equation", International Journal of Applied Mathematics, 32(03) (2019), 443-449.
- [15] Manju Somanath, J. Kannan and K. Raja, "On the Positive Integer Solutions for a Diophantine Equation", Journal of Mathematics and Informatics, 10 (2017), 173-177.
- [16] Manju Somanath and J. Kannan, "Congruum Problem", International Journal of Pure and Applied Mathematical Sciences (IJPAMS), 9(2) (2016), 123-131.



Impact Factor: 8.118



INTERNATIONAL JOURNAL OF INNOVATIVE RESEARCH

IN SCIENCE | ENGINEERING | TECHNOLOGY

 **9940 572 462**  **6381 907 438**  **ijirset@gmail.com**



www.ijirset.com

Scan to save the contact details