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Vector Space- A Mathematical Mini Review

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ABSTRACT: In mathematics and physics, a **vector space** (also called a **linear space**) is a set whose elements, often called vectors, may be added together and multiplied ("scaled") by numbers called scalars. Scalars are often real numbers, but can be complex numbers or, more generally, elements of any field. The operations of vector addition and scalar multiplication must satisfy certain requirements, called vector axioms. The terms **real vector space** and **complex vector space** are often used to specify the nature of the scalars: real coordinate space or complex coordinate space.

Vector spaces generalize Euclidean vectors, which allow modeling of physical quantities, such as forces and velocity, that have not only a magnitude, but also a direction. The concept of vector spaces is fundamental for linear algebra, together with the concept of matrix, which allows computing in vector spaces. This provides a concise and synthetic way for manipulating and studying systems of linear equations.

Vector spaces are characterized by their dimension, which, roughly speaking, specifies the number of independent directions in the space. This means that, for two vector spaces with the same dimension, the properties that depend only on the vector-space structure are exactly the same (technically the vector spaces are isomorphic). A vector space is finite-dimensional if its dimension is a natural number. Otherwise, it is infinite-dimensional, and its dimension is an infinite cardinal. Finite-dimensional vector spaces occur naturally in geometry and related areas. Infinite-dimensional vector spaces occur in many areas of mathematics. For example, polynomial rings are countably infinite-dimensional vector spaces, and many function spaces have the cardinality of the continuum as a dimension.

Many vector spaces that are considered in mathematics are also endowed with other structures. This is the case of algebras, which include field extensions, polynomial rings, associative algebras and Lie algebras. This is also the case of topological vector spaces, which include function spaces, inner product spaces, normed spaces, Hilbert spaces and Banach spaces.

KEYWORDS: vector space, complex, real, velocity, linear equations, dimension, algebras, polynomial

I. INTRODUCTION

In this article, vectors are represented in boldface to distinguish them from scalars.^[1]

A vector space over a field F is a set V together with two binary operations that satisfy the eight axioms listed below. In this context, the elements of V are commonly called vectors, and the elements of F are called scalars.

- The first operation, called vector addition or simply addition assigns to any two vectors \mathbf{v} and \mathbf{w} in V a third vector in V which is commonly written as $\mathbf{v} + \mathbf{w}$, and called the sum of these two vectors.
- The second operation, called scalar multiplication, assigns to any scalar a in F and any vector \mathbf{v} in V another vector in V , which is denoted $a\mathbf{v}$.^{1,2}

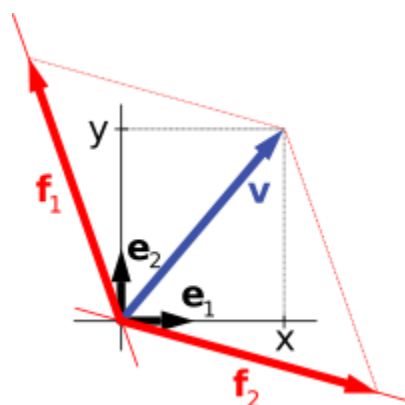
For having a vector space, the eight following axioms must be satisfied for every \mathbf{u} , \mathbf{v} and \mathbf{w} in V , and a and b in F .^[1]

Axiom	
Associativity of vector addition	$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
Commutativity of vector addition	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

Identity element of vector addition	There exists an element $\mathbf{0} \in V$, called the zero vector, such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.
Inverse elements of vector addition	For every $\mathbf{v} \in V$, there exists an element $-\mathbf{v} \in V$, called the additive inverse of \mathbf{v} , such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
Compatibility of scalar multiplication with field multiplication	$a(b\mathbf{v}) = (ab)\mathbf{v}$ [nb 3]
Identity element of scalar multiplication	$1\mathbf{v} = \mathbf{v}$, where 1 denotes the multiplicative identity in F .
Distributivity of scalar multiplication with respect to vector addition	$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
Distributivity of scalar multiplication with respect to field addition	$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$

When the scalar field is the real numbers the vector space is called a real vector space. When the scalar field is the complex numbers, the vector space is called a complex vector space. These two cases are the most common ones, but vector spaces with scalars in an arbitrary field F are also commonly considered. Such a vector space is called an F -vector space or a vector space over F .

An equivalent definition of a vector space can be given, which is much more concise but less elementary: the first four axioms (related to vector addition) say that a vector space is an abelian group under addition, and the four remaining axioms (related to the scalar multiplication), say that this operation defines a ring homomorphism from the field F into the endomorphism ring of this group.^{3,4}



A vector \mathbf{v} in \mathbb{R}^2 (blue) expressed in terms of different bases: using the standard basis of \mathbb{R}^2 : $\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2$ (black), and using a different, non-orthogonal basis: $\mathbf{v} = f_1 + f_2$ (red).

Linear independence

The elements of a subset G of a F -vector space V are said to be linearly independent if no element of G can be written as a linear combination of the other elements of G . Equivalently, they are linearly independent if two linear combinations of element of G define the same element of V if and only if they have the same coefficients. Also equivalently, they are linearly independent if a linear combination results in the zero vector if and only if all its coefficients are zero.^{5,6}

Linear subspace

A linear subspace or vector subspace W of a vector space V is a non-empty subset of V that is closed under vector addition and scalar multiplication; that is, the sum of two elements of W and the product of an element of V by a scalar belong to W . This implies that every linear combination of elements of W belongs to W . A linear subspace is a vector space for the induced addition and scalar multiplication; this means that the closure property implies that the axioms of a vector space are satisfied.

The closure property also implies that every intersection of linear subspaces is a linear subspace.

Linear span

Given a subset G of a vector space V , the linear span or simply the span of G is the smallest linear subspace of V that contains G , in the sense that it is the intersection of all linear subspaces that contain G . The span of G is also the set of all linear combinations of elements of G .

If W is the span of G , one says that G spans or generates W , and that G is a spanning set or a generating set of W .^{7,8}

Basis and dimension

A subset of a vector space is a basis if its elements are linearly independent and span the vector space. Every vector space has at least one basis, generally many (see Basis (linear algebra) § Proof that every vector space has a basis). Moreover, all bases of a vector space have the same cardinality, which is called the dimension of the vector space (see Dimension theorem for vector spaces). This is a fundamental property of vector spaces, which is detailed in the remainder of the section.

Bases are a fundamental tool for the study of vector spaces, especially when the dimension is finite. In the infinite-dimensional case, the existence of infinite bases, often called Hamel bases, depend on the axiom of choice. It follows that, in general, no base can be explicitly described. For example, the real numbers form an infinite-dimensional vector space over the rational numbers, for which no specific basis is known.^{9,10}

The one-to-one correspondence between vectors and their coordinate vectors maps vector addition to vector addition and scalar multiplication to scalar multiplication. It is thus a vector space isomorphism, which allows translating reasonings and computations on vectors into reasonings and computations on their coordinates. If, in turn, these coordinates are arranged as matrices, these reasonings and computations on coordinates can be expressed concisely as reasonings and computations on matrices. Moreover, a linear equation relating matrices can be expanded into a system of linear equations, and, conversely, every such system can be compacted into a linear equation on matrices.

So, in summary, finite-dimensional linear algebra may be expressed in three equivalent languages:

- Vector spaces, which provide concise and coordinate-free statements,
- Matrices, which are convenient for expressing concisely explicit computations,
- Systems of linear equations, which provide more elementary formulations.¹¹

II. DISCUSSION

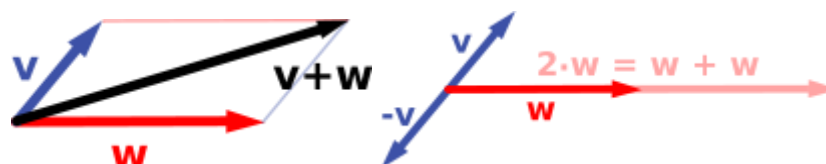
Vector spaces stem from affine geometry, via the introduction of coordinates in the plane or three-dimensional space. Around 1636, French mathematicians René Descartes and Pierre de Fermat founded analytic geometry by identifying solutions to an equation of two variables with points on a plane curve.^[2] To achieve geometric solutions without using coordinates, Bolzano introduced, in 1804, certain operations on points, lines and planes, which are predecessors of vectors.^[3] Möbius (1827) introduced the notion of barycentric coordinates. Bellavitis (1833) introduced the notion of a bipoint, i.e., an oriented segment one of whose ends is the origin and the other one a target.^[4] Vectors were reconsidered with the presentation of complex numbers by Argand and Hamilton and the inception of quaternions by

the latter.^[5] They are elements in \mathbf{R}^2 and \mathbf{R}^4 ; treating them using linear combinations goes back to Laguerre in 1867, who also defined systems of linear equations.^[12]

In 1857, Cayley introduced the matrix notation which allows for a harmonization and simplification of linear maps. Around the same time, Grassmann studied the barycentric calculus initiated by Möbius. He envisaged sets of abstract objects endowed with operations.^[6] In his work, the concepts of linear independence and dimension, as well as scalar products are present. Actually Grassmann's 1844 work exceeds the framework of vector spaces, since his considering multiplication, too, led him to what are today called algebras. Italian mathematician Peano was the first to give the modern definition of vector spaces and linear maps in 1888,^[7] although he called them "linear systems".^[8]

An important development of vector spaces is due to the construction of function spaces by Henri Lebesgue. This was later formalized by Banach and Hilbert, around 1920.^[9] At that time, algebra and the new field of functional analysis began to interact, notably with key concepts such as spaces of p-integrable functions and Hilbert spaces.^[10] Also at this time, the first studies concerning infinite-dimensional vector spaces were done.^[13]

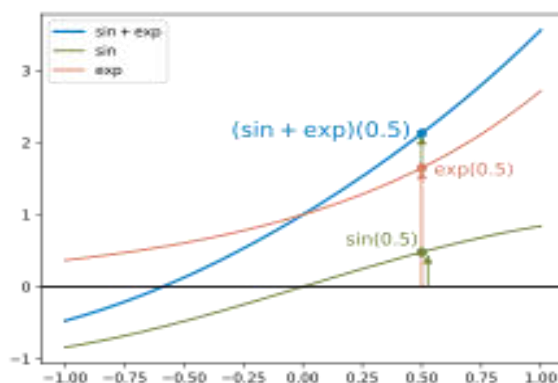
The first example of a vector space consists of arrows in a fixed plane, starting at one fixed point. This is used in physics to describe forces or velocities. Given any two such arrows, \mathbf{v} and \mathbf{w} , the parallelogram spanned by these two arrows contains one diagonal arrow that starts at the origin, too. This new arrow is called the sum of the two arrows, and is denoted $\mathbf{v} + \mathbf{w}$. In the special case of two arrows on the same line, their sum is the arrow on this line whose length is the sum or the difference of the lengths, depending on whether the arrows have the same direction. Another operation that can be done with arrows is scaling: given any positive real number a , the arrow that has the same direction as \mathbf{v} , but is dilated or shrunk by multiplying its length by a , is called multiplication of \mathbf{v} by a . It is denoted $a\mathbf{v}$. When a is negative, $a\mathbf{v}$ is defined as the arrow pointing in the opposite direction instead.



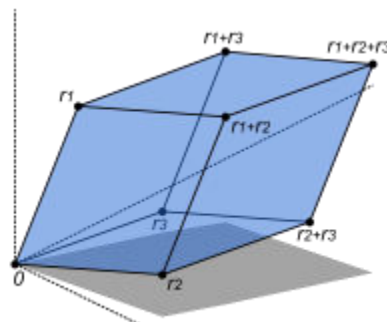
The set of complex numbers \mathbf{C} , that is, numbers that can be written in the form $x + iy$ for real numbers x and y where i is the imaginary unit, form a vector space over the reals with the usual addition and multiplication: $(x + iy) + (a + ib) = (x + a) + i(y + b)$ and $c \cdot (x + iy) = (c \cdot x) + i(c \cdot y)$ for real numbers x, y, a, b and c . The various axioms of a vector space follow from the fact that the same rules hold for complex number arithmetic.^[14]

In fact, the example of complex numbers is essentially the same as (that is, it is isomorphic to) the vector space of ordered pairs of real numbers mentioned above: if we think of the complex number $x + iy$ as representing the ordered pair (x, y) in the complex plane then we see that the rules for addition and scalar multiplication correspond exactly to those in the earlier example.^[11]

More generally, field extensions provide another class of examples of vector spaces, particularly in algebra and algebraic number theory: a field F containing a smaller field E is an E -vector space, by the given multiplication and addition operations of F .^[12]



Functions from any fixed set Ω to a field F also form vector spaces, by performing addition and scalar multiplication pointwise¹⁰

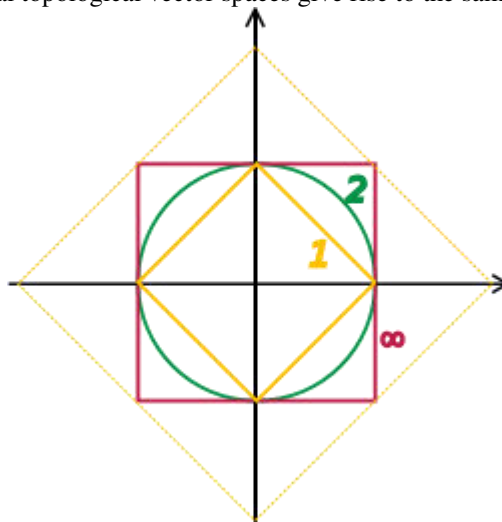


The volume of this parallelepiped is the absolute value of the determinant of the 3-by-3 matrix formed by the vectors r_1 , r_2 , and r_3 .⁹

III. RESULTS

From the point of view of linear algebra, vector spaces are completely understood insofar as any vector space is characterized, up to isomorphism, by its dimension. However, vector spaces per se do not offer a framework to deal with the question—crucial to analysis—whether a sequence of functions converges to another function. Likewise, linear algebra is not adapted to deal with infinite series, since the addition operation allows only finitely many terms to be added. Therefore, the needs of functional analysis require considering additional structures.⁸

Banach and Hilbert spaces are complete topological vector spaces whose topologies are given, respectively, by a norm and an inner product. Their study—a key piece of functional analysis—focuses on infinite-dimensional vector spaces, since all norms on finite-dimensional topological vector spaces give rise to the same notion of convergence⁷



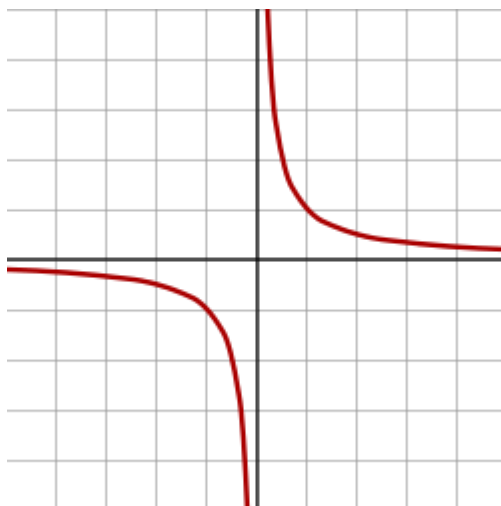
Unit "spheres" in R^2 consist of plane vectors of norm 1. Depicted are the unit spheres in different p -norms, for $p=1, 2$ & ∞ . The bigger diamond depicts points of 1-norm equal to 2.

The solutions to various differential equations can be interpreted in terms of Hilbert spaces. For example, a great many fields in physics and engineering lead to such equations and frequently solutions with particular physical properties are used as basis functions, often orthogonal.^[56] As an example from physics, the time-dependent Schrödinger equation in quantum mechanics describes the change of physical properties in time by means of a partial differential equation, whose solutions are called wavefunctions. Definite values for physical properties such as energy, or momentum, correspond to eigenvalues of a certain (linear) differential operator and the associated wavefunctions are called eigenstates. The spectral theorem decomposes a linear compact operator acting on functions in terms of these eigenfunctions and their eigenvalues.^{5,6}

General vector spaces do not possess a multiplication between vectors. A vector space equipped with an additional bilinear operator defining the multiplication of two vectors is an algebra over a field.^[59] Many algebras stem from functions on some geometrical object: since functions with values in a given field can be multiplied pointwise,

these entities form algebras. The Stone–Weierstrass theorem, for example, relies on Banach algebras which are both Banach spaces and algebras.

Commutative algebra makes great use of rings of polynomials in one or several variables, introduced above. Their multiplication is both commutative and associative. These rings and their quotients form the basis of algebraic geometry, because they are rings of functions of algebraic geometric objects.^{3,4}



A hyperbola given by equation $x,y=1$

IV. CONCLUSIONS

Vectors in specific vector spaces

- Column vector, a matrix with only one column. The column vectors with a fixed number of rows form a vector space.
- Row vector, a matrix with only one row. The row vectors with a fixed number of columns form a vector space.
- Coordinate vector, the n -tuple of the coordinates of a vector on a basis of n elements. For a vector space over

a field F , these n -tuples form the vector space (where the operation are pointwise addition and scalar multiplication).^{2,3}

- Displacement vector, a vector that specifies the change in position of a point relative to a previous position. Displacement vectors belong to the vector space of translations.
- Position vector of a point, the displacement vector from a reference point (called the origin) to the point. A position vector represents the position of a point in a Euclidean space or an affine space.
- Velocity vector, the derivative, with respect to time, of the position vector. It does not depend of the choice of the origin, and, thus belongs to the vector space of translations.
- Pseudovector, also called axial vector
- Covector, an element of the dual of a vector space. In an inner product space, the inner product defines an isomorphism between the space and its dual, which may make difficult to distinguish a covector from a vector. The distinction becomes apparent when one changes coordinates (non-orthogonally).
- Tangent vector, an element of the tangent space of a curve, a surface or, more generally, a differential manifold at a given point (these tangent spaces are naturally endowed with a structure of vector space)
- Normal vector or simply normal, in a Euclidean space or, more generally, in an inner product space, a vector that is perpendicular to a tangent space at a point.
- Gradient, the coordinates vector of the partial derivatives of a function of several real variables. In a Euclidean space the gradient gives the magnitude and direction of maximum increase of a scalar field. The gradient is a covector that is normal to a level curve.
- Four-vector, in the theory of relativity, a vector in a four-dimensional real vector space called Minkowski space^{1,2}

REFERENCES

1. It is also common, especially in physics, to denote vectors with an arrow on top: \vec{v} . It is also common, especially in higher mathematics, to not use any typographical method for distinguishing vectors from other mathematical objects.
2. \wedge Scalar multiplication is not to be confused with the scalar product, which is an additional operation on some specific vector spaces, called inner product spaces. Scalar multiplication is a multiplication of a vector by a scalar that produces a vector, while the scalar product is a multiplication of two vectors that produces a scalar.
3. \wedge This axiom is not an associative property, since it refers to two different operations, scalar multiplication and field multiplication. So, it is independent from the associativity of field multiplication, which is assumed by field axioms.
4. \wedge The nomenclature derives from German "eigen", which means own or proper.
5. \wedge See also Jordan–Chevalley decomposition.
6. \wedge This is typically the case when a vector space is also considered as an affine space. In this case, a linear subspace contains the zero vector, while an affine subspace does not necessarily contain it.
7. \wedge Some authors (such as Roman 2005) choose to start with this equivalence relation and derive the concrete shape of \sim from this.
8. \wedge This requirement implies that the topology gives rise to a uniform structure, Bourbaki 1989ch. II ,
9. \wedge The triangle inequality for $\| \cdot \|$ is provided by the Minkowski inequality. For technical reasons, in the context of functions one has to identify functions that agree almost everywhere to get a norm, and not only a seminorm.
10. \wedge "Many functions in L^1 of Lebesgue measure, being unbounded, cannot be integrated with the classical Riemann integral. So spaces of Riemann integrable functions would not be complete in the norm, and the orthogonal decomposition would not apply to them. This shows one of the advantages of Lebesgue integration.", Dudley 1989§5.3, p. 125 ,
11. \wedge For ℓ^2 is not a Hilbert space.
12. \wedge A basis of a Hilbert space is not the same thing as a basis in the sense of linear algebra above. For distinction, the latter is then called a Hamel basis.
13. \wedge That is, there is a homeomorphism from $\pi^{-1}(U)$ to $V \times U$ which restricts to linear isomorphisms between fibers.
14. \wedge A line bundle, such as the tangent bundle of S^1 is trivial if and only if there is a section that vanishes nowhere, see Husemoller 1994Corollary 8.3. The sections of the tangent bundle are just \mathbb{R} -valued vector fields.



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