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Bilinear Optimal Control for Stochastic Heat Equations

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ABSTRACT: We consider the bilinear optimal control problem for stochastic heat equation with multiplicative controls. Optimal control problems for infinite dimensional stochastic equations has been studied by many researchers. Most of the researchers have considered the Hamilton-Jacobi-Bellman(HJB) approach and obtained the optimal feedback laws using associated HJB equation on a Hilbert space. In present work, we do not use HJB equations. Also, Riccati equation framework is not suitable due to nonlinear nature introduced by multiplicative controls. We prove the existence of optimal control and then give the characterization in the form of optimality system.

KEYWORDS: Optimal control; Stochastic heat equation; Additive noise; Bilinear control, Brownian motion.

I. INTRODUCTION

In this paper, we will consider the bilinear optimal control problem for stochastic heat equation with multiplicative controls. Optimal control problems for infinite dimensional stochastic equations have been studied by Barbu and Da Prato[1], Cannarsa and Da Prato[2, 3], Gozzi[4, 5]. In these papers, authors consider the Hamilton-Jacobi-Bellman(HJB) approach and obtain the optimal feedback law using associated HJB equation on a Hilbert space. In most of the papers backward stochastic differential equations(BSDEs) are applied to solve stochastic optimal control problems. For example, see [6, 7, 8] and references therein. The maximum principle has been studied in some of these papers to find necessary/sufficient conditions for optimality for stochastic partial differential(SPDEs). In present work, we do not use HJB equations. Also, Riccati equation framework is not suitable due to nonlinear nature introduced by multiplicative control. The paper is organized as follows. We will mention some preliminaries in section 2. We will prove the existence of optimal control in section 3 and obtain the optimality system in section 4.

II. PRELIMINARIES

Let $D \subset \mathbb{R}^d$ be a bounded domain with smooth boundary ∂D . Consider the linear stochastic heat equation defined by

$$dy(\xi, t) = [\Delta y(\xi, t) + y(\xi, t)u(\xi, t) + f(\xi, t)]dt + \sigma(\xi, t)dW(\xi, t), \quad (1)$$

for all $\xi \in D$ and all $t \in [0, T]$. f and σ are given adapted random fields which are bounded and continuous on $D \times [0, T]$ a.s. $W(\cdot, t)$ is an $L^2(D)$ -valued Q -Wiener process and $T \in (0, \infty)$ is a real number. The multiplicative control $u(\xi, t)$ is a bounded stochastic process. For the simplicity we neglect the explicit dependence on $\omega \in \Omega$ and homogeneous Dirichlet boundary conditions are chosen

$$\begin{aligned} y(\xi, 0) &= \psi(\xi) \in L^2(D), & \forall \xi \in D, \\ y(\xi, t) &= 0, & \forall (\xi, t) \in \partial D \times [0, T]. \end{aligned} \quad (2)$$

Let $H = K = L^2(D)$, $V = H_0^1(D)$, $V' = H^{-1}(D)$ and define

$$A\phi = \Delta\phi + u\phi.$$

We define a bilinear form

$$\langle A\phi, \chi \rangle = -\int_0^1 \nabla\phi \cdot \nabla\chi dx + \int_0^1 u\phi\chi dx.$$

Definition 1: - An H -valued predictable process $y(\cdot, t)$, $t \in [0, T]$ is said to be a weak solution of (1) if

1. the trajectories of $y(\cdot, t)$ are \mathbb{P} -a.s. Bochner integrable and
 2. for all $v(\cdot) \in D(A^*)$ and all $t \in [0, T]$ we have
- $$\langle y(\cdot, t), v(\cdot) \rangle = \langle \psi(\cdot), v(\cdot) \rangle + \int_0^t [\langle Ay(\cdot, s), v(\cdot) \rangle + \langle f(\cdot, s), v(\cdot) \rangle] ds + \langle W(\cdot, t), v(\cdot) \rangle, \quad \mathbb{P} - \text{a. s.} \quad (3)$$



Let $(\lambda_k)_{k \in \mathbb{N}}$ be the eigenvalues of the covariance operator Q and $(\mu_k)_{k \in \mathbb{N}}$ be the eigenvalues of the Laplace operator Δ with homogeneous boundary conditions. If

$$\sum_{k=1}^{\infty} \lambda_k \mu_k^2 < \infty$$

then the trajectories of the process $y(\cdot, t)$ are in the Sobolev space $H_0^1(D)$.

The $L^2(D)$ -valued Q -wiener process is represented by

$$W(\xi, t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} e_k(t) \beta_k(t), \quad \forall t \in [0, T] \text{ and all } \xi \in D, \tag{4}$$

where $(\beta_k(t), t \in [0, T])$ with $\{k \in \mathbb{N} : \lambda_k > 0\}$ are independent real-valued Brownian motions on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\{e_k\}_{k \in \mathbb{N}}$ is an orthonormal basis of $L^2(D)$ consisting of the eigenvectors of the covariance operator Q with the corresponding eigenvalues $\{\lambda_k\}_{k \in \mathbb{N}}$ such that

$$Q e_k(\cdot) = \lambda_k e_k(\cdot), \quad \forall k \in \mathbb{N}, \lambda_k \geq 0,$$

and zero is the only accumulation point of the sequence $\{\lambda_k\}_{k \in \mathbb{N}}$. The series (4) converges in $L^2(\Omega, \mathcal{F}, \mathbb{P}; C([0, T]; H))$, and thus has a \mathbb{P} -a.s. continuous modification[9].

Assume that

$$\mathbb{E} \int_0^T \left\{ \int_D |f(\xi, t)|^2 dx + \int_D Q \sigma^2(\xi, t) d\xi \right\} dt < \infty.$$

Then, by Theorem 7.2 from [10], there exist the unique weak solution y of (1) which belongs to $L^2(\Omega; C([0, T]; H)) \cap L^2(\Omega \times [0, T]; V)$. Moreover, using Lemma 7.3 from [10], we have the following estimate for a weak solution y of (1)

$$\mathbb{E} \sup_{0 \leq t \leq T} \|y\|^2 + \mathbb{E} \int_0^T \|y\|_V^2 ds \leq C_T \left\{ \|\psi\|^2 + \mathbb{E} \int_0^T (\|f\|^2 + \|\sigma\|_Q^2) ds \right\} \tag{5}$$

for some constant $C_T > 0$.

We consider the optimization problem

$$J(u) = \mathbb{E} \left(\frac{1}{2} \int_0^t \int_D |y - z|^2 d\xi dt + \frac{\beta}{2} \int_0^t \int_D |u|^2 d\xi dt \right) \tag{6}$$

constrained by the stochastic heat equation (1) with initial and Dirichlet boundary conditions (2). z in (6) is a desired target solution and β is a positive constant. Our aim is to minimize the the functional J over the set \mathcal{U} of all admissible controls defined by

$$\mathcal{U} = \{u \in L^\infty(\Omega; L^\infty(D \times [0, T])) : \exists y \in L^2(\Omega; C([0, T]; H)) \text{ corresponding to satisfying (1), (12) and } J(y, u) < \infty\}.$$

III. EXISTENCE OF AN OPTIMAL CONTROL

In this section we will prove the existence of an optimal control.

Lemma 1: - Let $\{u_n\} \subset \mathcal{U}$ be a weakly convergent sequence converging to $u \in \mathcal{U}$ and let y_n be a weak solution corresponding u_n of the stochastic heat equation

$$dy_n(\xi, t) = [\Delta y_n(\xi, t) + u_n(\xi, t) y_n(\xi, t) + f(\xi, t)] dt + \sigma(\xi, t) dW(\xi, t)$$

with initial and boundary conditions as in (2). Then the sequence $\{y_n\}$ in $L^2(\Omega; C([0, T]; H))$ converges weakly to the solution y of the stochastic heat equation (1) with control u and the initial and boundary conditions (2).

Proof. From inequality (5) it follows that $\{y_n(\cdot, t)\}$ is a bounded sequence in $L^2(\Omega \times [0, T])$. By Rellich embedding theorem it follows that $\{y_n\}$ is a Cauchy sequence in $L^2(\Omega; C([0, T]; H))$. As $L^2(\Omega; C([0, T]; H))$ is complete space, $\{y_n\}$ converges weakly to a process $y \in L^2(\Omega; C([0, T]; H))$. Also, $\{u_n\}$ converges weakly to u , and hence $\{y_n u_n\}$ converges weakly to yu . Hence due to weak formulation (3) it follows that y is a weak solution of (1) with initial and boundary conditions (2)

Theorem 1: - There exists an optimal control $\hat{u} \in \mathcal{U}$ which minimizes the cost functional (6).

Proof. Let $\{u^n\} \subset \mathcal{U}$ be a minimizing sequence such that

$$\lim_{n \rightarrow \infty} J(u^n) = \inf_{u \in \mathcal{U}} J(u).$$

As $\{u^n\}$ is a bounded sequence in a Banach space $L^\infty(\Omega; L^\infty(D \times [0, T]))$, there exists a subsequence of $\{u^n\}$, (for simplicity) again denoted by $\{u^n\}$ such that $\{u^n\}$ is weakly convergent in $L^\infty(\Omega; L^\infty(D \times [0, T]))$. Let \hat{u} be a limit of $\{u^n\}$. Further, let y^n be a weak solution of the stochastic heat equation (1) and (2) corresponding to the control u^n . Then by Lemma 1, there exist a weak solution \hat{y} of the stochastic heat equation (1) and (2) corresponding to \hat{u} . Hence $\hat{u} \in \mathcal{U}$. By the weak lower semi-continuity of the cost functional J we have

$$J(\hat{u}) \leq \lim_{n \rightarrow \infty} J(u^n) = \inf_{u \in \mathcal{U}} J(u).$$

Thus \hat{u} is an optimal control.

IV. THE OPTIMALITY SYSTEM

To obtain the optimality system, we require the adjoint problem of the stochastic heat problem (1) with initial and boundary conditions (2). In general, we want to find a function $\Phi: \Omega \times D \times [0, T] \rightarrow \mathbb{R}$ with

$$\mathbb{E} \int_0^T \int_0^1 |\Phi(\xi, t)|^2 d\xi dt < \infty$$

such that

$$\mathbb{E} \int_0^T \int_D \Phi(\xi, t) \left[\frac{\partial}{\partial t} \eta(\xi, t) - A\eta(\xi, t) \right] d\xi dt = \mathbb{E} \int_0^T \int_D (y - z)(\xi, t) \eta(\xi, t) d\xi dt. \tag{7}$$

for all $\eta: \Omega \times D \times [0, T] \rightarrow \mathbb{R}$ and all admissible controls $u \in \mathcal{U}$, where $y(\xi, t)$ is the weak solution of the stochastic heat problem (1), (2) and $z(\xi, t) \in L^2(\Omega; C([0, T]; H))$ is desired profile given in (6).

The solution $\Phi(\xi, t)$ of equation (7) is called the generalized solution of the adjoint problem of the stochastic heat problem (1), (2), where the adjoint equation is

$$-d\Phi(\xi, t) = A\Phi(\xi, t)dt + (y - z)(\xi, t)dt \tag{8}$$

for all $\xi \in D$ and all $t \in [0, T]$, with the terminal and Dirichlet boundary conditions

$$\begin{aligned} \Phi(\xi, T) &= 0, & \forall \xi \in [0, 1], \\ \Phi(\xi, t) &= 0, & \forall (\xi, t) \in \partial D \times [0, t]. \end{aligned} \tag{9}$$

The adjoint problem is the deterministic heat problem with random variable. The existence and uniqueness of the unique generalized solution to the adjoint problem follows from the deterministic case. Furthermore, for the generalized solution $\Phi(\xi, t)$ of the adjoint problem, there exists a positive constant C such that

$$\|\Phi(\cdot, t)\|_H^2 \leq C, \quad \|\Phi(\cdot, t)\|_V^2 \leq C, \quad \mathbb{P} - a. s. \quad \forall t \in [0, T]. \tag{10}$$

By y_u , we will mean a solution y of (1), (2) corresponding to the arbitrary control u . Let u be an admissible control and $\epsilon > 0$ be sufficiently small such that $u + \epsilon l$ is also an admissible control. Let δy be the difference between $y_{u+\epsilon l}$ and y_u , that is, $\delta y = y_{u+\epsilon l} - y_u$.

Lemma 2: - δy is the weak solution of the initial and boundary value problem

$$\begin{aligned} \frac{\partial}{\partial t} \delta y(\xi, t) &= \Delta \delta y + (u + \epsilon l) \delta y + \epsilon l y_u, & \forall (\xi, t) \in D \times (0, T), \\ \delta y(\xi, 0) &= 0, & \forall \xi \in D, \\ \delta y(\xi, t) &= 0, & \forall (\xi, t) \in \partial D \times [0, T]. \end{aligned} \tag{11}$$

Proof. By taking two solutions of this kind and taking their difference we get the system (11).

Theorem 2: - The Gâteaux derivative of the objective functional (6) is represented by

$$\delta J(u)(l) = \mathbb{E} \int_0^T \int_D l(\beta u + \Phi y_u) \tag{12}$$

for all admissible controls $l \in \mathcal{U}$.

Proof. We have

$$\begin{aligned} J(u + \epsilon l) + J(u) &= \mathbb{E} \left(\frac{1}{2} \int_0^T \int_D (y_{u+\epsilon l} - z)^2 d\xi dt \right) + \mathbb{E} \left(\frac{\beta}{2} \int_0^T \int_D (u + \epsilon l)^2 d\xi dt \right) \\ &\quad - \mathbb{E} \left(\frac{1}{2} \int_0^T \int_D (y_u - z)^2 d\xi dt \right) - \mathbb{E} \left(\frac{\beta}{2} \int_0^T \int_D u^2 d\xi dt \right), \end{aligned}$$

which simplifies to

$$\begin{aligned} J(u + \epsilon l) - J(u) &= \mathbb{E} \left(\frac{1}{2} \int_0^T \int_D \delta y (y_{u+\epsilon l} + y_u - 2z) d\xi dt \right) \\ &\quad + \mathbb{E} \left(\frac{\beta}{2} \int_0^T \int_D (2u\epsilon l + \epsilon^2 l^2) d\xi dt \right). \end{aligned} \tag{13}$$

Replacing η by δy in (7) gives

$$\mathbb{E} \int_0^T \int_D \Phi \left[\frac{\partial}{\partial t} \delta y - \Delta \delta y - u \delta y \right] d\xi dt = \mathbb{E} \int_0^T \int_D (y - z) \delta y d\xi dt. \tag{14}$$

From (11), we obtain

$$\mathbb{E} \int_0^T \int_D \Phi \left[-\frac{\partial}{\partial t} \delta y + \Delta \delta y + (u + \epsilon l) \delta y \right] d\xi dt = -\mathbb{E} \int_0^T \int_D \Phi \epsilon l y_u d\xi dt. \tag{15}$$

Adding (14) and (15), we have

$$\mathbb{E} \int_0^T \int_D \Phi \epsilon l \delta y d\xi dt = \mathbb{E} \int_0^T \int_D (y - z) \delta y d\xi dt - \mathbb{E} \int_0^T \int_D \Phi \epsilon l y_u d\xi dt.$$

Thus

$$\mathbb{E} \int_0^T \int_D (y - z) \delta y d\xi dt = \mathbb{E} \left(\int_0^T \int_D [\Phi \epsilon l \delta y + \Phi \epsilon l y_u] d\xi dt \right).$$

Hence



$$\begin{aligned} \mathbb{E} \int_0^T \int_D \lim_{\epsilon \rightarrow 0} (y - z) \frac{\delta y}{\epsilon} d\xi dt &= \lim_{\epsilon \rightarrow 0} \mathbb{E} \int_0^T \int_D (y - z) \frac{\delta y}{\epsilon} d\xi \\ &= \mathbb{E} \int_0^T \int_D \Phi l y_u d\xi dt. \end{aligned}$$

Now,

$$\begin{aligned} \delta J(u)(l) &= \lim_{\epsilon \rightarrow 0} \frac{J(u+\epsilon l) - J(u)}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \mathbb{E} \left(\frac{1}{2} \int_0^T \int_D \frac{\delta y(y_{u+\epsilon l} + y_u - 2z)}{\epsilon} d\xi dt + \frac{\beta}{2} \int_0^T \int_D (2ul + \epsilon l^2) d\xi dt \right) \\ &= \mathbb{E} \left(\int_0^T \int_D \lim_{\epsilon \rightarrow 0} \frac{\delta y}{\epsilon} (y - z) d\xi dt \right) + \mathbb{E} \int_0^T \int_D \beta ul d\xi dt. \end{aligned}$$

So,

$$\begin{aligned} \delta J(u)(l) &= \mathbb{E} \int_0^T \int_D l y_u \Phi d\xi dt + \mathbb{E} \int_0^T \int_D \beta ul d\xi dt \\ &= \mathbb{E} \int_0^T \int_D l (y_u \Phi + \beta u) d\xi dt. \end{aligned} \tag{16}$$

Hence, the necessary optimality condition

$$\mathbb{E}(\beta u + \Phi y_u) \geq 0 \tag{17}$$

can be formulated.

By standard control argument, from inequality (17) we get

$$\beta u + \Phi y_u = 0 \quad \mathbb{P} - \text{a. s.} \tag{18}$$

The stochastic heat equation (1), (2), adjoint equation (8), (9) and equality (18) forms the optimality system (OS).

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