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On Some Probabilistic Properties of the Law of Facility of Errors

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ABSTRACT: The errors that occur at any time during the process or phenomenon may be random. These random errors can be defined mathematically by the law of facility errors. The integrals involving this error function occur in various fields of science. Here, an attempt is made to study some of the probabilistic properties of the law of facility of errors, such as mean, variance, moment generating function, etc. The parameter involved in the function is estimated using the method of maximum likelihood. It is seen that the values of the true parameter along with the confidence interval are properly estimated using the method of maximum likelihood.

KEYWORDS: Law of facility of error, maximum likelihood estimator, Confidence Interval, Information, mean sum of squares.

I.INTRODUCTION

Today's research in science, engineering, artificial intelligence, robotics, and other related areas belongs to the world of information. The basis for this information is a large amount of data and its measurement. The data and its measurement form a strong foundation for developments in the above fields. Measurement of any data is an essential part of every field. The measurement of the data must have happened with accuracy. Sometimes there may be chances that the measurement of the data may have deviated from the actual values. The deviation from actual values may be termed an error. The reliability of the measurement of data may be compromised due to plausible errors by the machine or human beings, nature or the environment, etc.

The errors made by human beings are referred to as human errors or gross errors. The errors incurred by the machines or the incorrect processing environment are termed systematic errors. The gross errors and systematic errors can be compiled and resolved during the measurement process. The errors that may occur at any time during the process or experiments at irregular magnitudes are known as accidental errors or random errors.

The occurrence of random errors may follow some law or mathematical function. For several numbers of errors, the most probable value of the error is the average of the errors. This average of the errors is the closest approximation to the expected value calculated using the method of least squares. The standard deviations of these errors provide the dispersion around the average.

This presented article considers that random errors follow the law of errors of observations [2] and the law of facility of errors [3], [4], and [6]. Integrals of these error functions can occur in the theory of measurements, mathematical physics, atomic physics, astrophysics, diffusion theory, mathematical models in biology, chemical analysis, statistical analysis, etc.

According to Crofton (1860), the probability of an error being found to lie between x and $x + \delta x$ is given by

$$\Phi(x) = \frac{1}{\sqrt{2\pi(h-i)}} \exp\left(-\frac{(x-m)^2}{2(h-i)}\right) dx \quad (1)$$

where,

$m = \text{sum of mean errors}$

$h = \text{sum of mean squares of errors}$

$i = \text{sum of squares of mean errors}$

If the occurrence of errors has two equally likely forms, i.e. positive and negative, in the observations, then the sum of mean errors is zero, i.e. $m = 0$. Then $\Phi(x)$ becomes



$$\Phi(x) = \frac{1}{c\sqrt{\pi}} \exp\left(-\frac{x^2}{c^2}\right) \tag{2}$$

where,

$$c = 2(h - i)$$

A similar function had been introduced by Glaisher (1872) as the law of facility of errors.

$$f(x) = \frac{\theta}{\sqrt{\pi}} \exp(-\theta^2 x^2) \tag{3}$$

Glaisher (1872) inferred from eq. (3) that the arithmetic mean is the most probable result. In this paper, the eq. (3) is introduced as a probability distribution for the errors that occurred during software testing. It satisfy the axioms of probability for $\theta > 0$ such that

$$\frac{\theta}{\sqrt{\pi}} \exp(-\theta^2 x^2) > 0 \forall -\infty \leq X \leq \infty, \theta > 0$$

and

$$\int_{-\infty}^{\infty} \frac{\theta}{\sqrt{\pi}} \exp(-\theta^2 x^2) dx = \frac{\theta}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{\theta} = 1; \forall -\infty \leq X \leq \infty, \theta > 0 \tag{4}$$

Hence, if the random variable X is said to follow the law of facility of errors if its probability density function is given by

$$f(x) = \begin{cases} \frac{\theta}{\sqrt{\pi}} \exp(-\theta^2 x^2) & ; -\infty \leq X \leq \infty, \theta > 0 \\ 0 & ; otherwise \end{cases} \tag{5}$$

The distribution function of is

$$F(x) = \begin{cases} \int_{-\infty}^u \frac{\theta}{\sqrt{\pi}} \exp(-\theta^2 x^2) dx & ; -\infty \leq X \leq \infty, \theta > 0 \\ 0 & ; otherwise \end{cases} \tag{6}$$

The solution to equation (6) is obtained in the following equations (7) and (8) using [1], [7]-[12], and [16]-[17].

$$F(x) = \begin{cases} \frac{1}{2} + \frac{1}{2} \Phi(\theta u) & ; -\infty \leq X \leq \infty, \theta > 0 \\ 0 & ; otherwise \end{cases} \tag{7}$$

where,

$\Phi(ax)$ is an entire function of the complex variable x is called a probability integral. At zero, we have $\Phi(0) = 0$ and as x increases, $\Phi(\theta x)$ rapidly approaches the limiting value $\Phi(\infty) = 1$ since it satisfies equation (4).

Equation (6) can also be represented in the Confluent Hyper-geometric function as

$$F(x) = \begin{cases} \frac{1}{2} + \frac{1}{2} {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -\theta^2 u^2\right) & ; -\infty \leq X \leq \infty, \theta > 0 \\ 0 & ; otherwise \end{cases} \tag{8}$$

Properties of the distribution:

Property	Formula	Value of Property
Mean (μ)	$\frac{\theta}{\sqrt{\pi}} \int_{-\infty}^{\infty} x \exp(-\theta^2 x^2)$	0
Variance μ_2	$\frac{\theta}{\sqrt{\pi}} \int_{-\infty}^{\infty} (x - \mu)^2 \exp(-\theta^2 x^2)$	$\frac{1}{2\theta^2}$
Mode	value of x related to $f'(x) = 0$	0
Median		0
Odd ordered Raw moments μ'_{2r+1}	$\frac{\theta}{\sqrt{\pi}} \int_{-\infty}^{\infty} x^{2r+1} \exp(-\theta^2 x^2)$	0
Even ordered Raw moments μ'_{2r}	$\frac{\theta}{\sqrt{\pi}} \int_{-\infty}^{\infty} x^{2r} \exp(-\theta^2 x^2)$	$\frac{\theta \Gamma\left(r + \frac{1}{2}\right)}{\sqrt{\pi}(\theta^2)^{\left(r + \frac{1}{2}\right)}}$
Moment Generating Function	$M_X(t) = \frac{\theta}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{tx} \exp(-\theta^2 x^2)$	$\frac{\theta}{ \theta } e^{\frac{t^2}{4\theta^2}}$ or $e^{\frac{t^2}{4\theta^2}}$ if $\theta > 0$



Cumulant Generating Function	$K_X(t) = \log M_X(t)$	$\frac{t^2}{4\theta^2}$ if $\theta > 0$
Measures of Skewness and Kurtosis		$\beta_1 = 0, \beta_2 = 3, \gamma_1 = 0, \gamma_2 = 0$

Maximum Likelihood Estimation: Suppose x_1, x_2, \dots, x_n is a sample drawn from the above population defined by (5) then the likelihood function of the data is obtained as

$$L(\theta/X) = \prod_{i=1}^n f(x_i) = \prod_{i=1}^n \left[\frac{\theta}{\sqrt{\pi}} \exp(-\theta^2 x_i^2) \right]$$

The likelihood becomes

$$L(\theta/X) = \left(\frac{\theta}{\sqrt{\pi}} \right)^n \exp(-\sum_{i=1}^n \theta^2 x_i^2) \tag{9}$$

The log-likelihood is

$$\ln L(\theta/X) = n \ln \left(\frac{\theta}{\sqrt{\pi}} \right) - \sum_{i=1}^n \theta^2 x_i^2$$

The parameter θ can be estimated by solving, $\frac{\partial}{\partial \theta} \ln L(\theta/X) = 0$, and the estimator of θ can be obtained as

$$\hat{\theta} = \sqrt{\frac{n}{2T}} \tag{10}$$

Where, $T = \sum_{i=1}^n x_i^2$ $n \rightarrow \infty, \hat{\theta} \rightarrow \theta$ i.e. the MLE $\hat{\theta}$ approaches the true parameter θ , which is also known as the consistency property of the MLE.

Asymptotic Normal Approximation:

Consider a sample $x_1, x_2, \dots, x_n (= X)$ that is chosen at random from the population (5) with parameter θ . The term

$\frac{\partial}{\partial \theta} \ln L(\theta/X)$ is a score function and $\frac{1}{\sqrt{-\frac{\partial^2}{\partial \theta^2} \ln L(\theta/X)}}$ is an approximate standard error for $\hat{\theta}$.

The function $-\frac{\partial^2}{\partial \theta^2} \ln L(\theta/X)$ is observed information and $I(\theta) = -E \left[\frac{\partial^2}{\partial \theta^2} \ln L(\theta/X) \right]$ is expected Fisher information.

The Fisher information shows how much knowledge is available about the parameter θ in the random sample X . In statistical methodologies, both the Bayesian and frequentist approaches use Fisher information to seek the initial probability and to build the confidence interval for MLE. The confidence interval with confidence coefficient $1 - \alpha$ for the parameter θ using Fisher information is obtained as $\hat{\theta} \pm Z_{\frac{\alpha}{2}} \frac{1}{\sqrt{I(\hat{\theta})}}$.

In present case, using $I(\theta) = 4T$ and equation (10), the $(1 - \alpha)\%$ confidence interval is

$$\theta \in \left[\sqrt{\frac{n}{2T}} \pm Z_{\frac{\alpha}{2}} \frac{1}{\sqrt{4T}} \right] \text{ or } \theta \in \left[\frac{1}{2\sqrt{T}} \left(\sqrt{2n} \pm Z_{\frac{\alpha}{2}} \right) \right] \tag{11}$$

The lower bound for the variance of the MLE $\hat{\theta}$ can be obtained as $Var(\hat{\theta}) \geq \frac{1}{I(\theta)}$

Numerical Evaluation and Conclusions: For the numerical evaluation, the random numbers are generated using a

generalized gamma distribution with the particular values of parameters a, b , and c i.e. $g(x) = \frac{bx^{c-1}e^{-\left(\frac{x}{a}\right)^b}}{a^c \Gamma(c/b)}$; $X > 0, a, b, c > 0$. To generate the random numbers from a generalized gamma distribution, the particular values are considered as $a = \frac{1}{\theta}, b = 2$, and $c = 1$. The numerical evaluation is summarized in the following table.



Value of θ	Maximum Likelihood Estimator
0.005	0.004995378
0.05	0.04995899
0.5	0.4991179
5	4.992317
10	9.980383
50	50.00931

From the above numerical evaluation, it is seen that the true value of the parameter θ is estimated with high precision.

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