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On Secondary Orthogonal Bimatrices

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ABSTRACT: Secondary orthogonal bimatrices are studied as a generalization of orthogonal and secondary orthogonal matrices. Some of the properties of orthogonal and secondary orthogonal matrices are extended to secondary orthogonal bimatrices. Some important results of orthogonal and secondary orthogonal matrices are generalized to secondary orthogonal bimatrices.

KEYWORDS: Orthogonal matrix, secondary orthogonal matrix, bimatrix, symmetric bimatrix, skew-symmetric bimatrix, orthogonal bimatrix, secondary orthogonal bimatrix.

AMS classification: 15A09, 15A15, 15A57.

I. INTRODUCTION

Matrices provide a very powerful tool for dealing with linear models. Bimatrices are still a powerful and an advanced tool which can handle over one linear model at a time. Bimatrices are useful when time bound comparisons are needed in the analysis of a model. Bimatrices are of several types. We denote the space of $n \times n$ complex matrices by $\mathcal{C}_{n \times n}$. For $A \in \mathcal{C}_{n \times n}$, A^T , A^{-1} , A^\dagger and $\det(A)$ denote transpose, inverse, Moore-Penrose inverse and determinant of A respectively [3]. The secondary transpose A^S defined from $A = (a_{ij})$ by reflecting its entries through the secondary diagonal. If $A = (a_{ij})$, then $A^S = (b_{ij})$, where $b_{ij} = a_{n-j+1, n-i+1}$. The usual transpose A^T and A^S are related as $A^S = VA^TV$, where V is the permutation matrix with units in its secondary diagonal [2]. If $AA^T = A^TA = I$ then A is an orthogonal matrix and if $AA^S = A^SA = I$ then A is a secondary orthogonal matrix, where I is an identity matrix [1, 2]. In this paper, we study the secondary orthogonal bimatrices as a generalization of orthogonal and secondary orthogonal matrices. Some of the properties of orthogonal and secondary orthogonal matrices are extended to secondary orthogonal bimatrices. Some important results of orthogonal and secondary orthogonal matrices are generalized to secondary orthogonal bimatrices.

II. PRELIMINARIES

Definition 2.1 [5]

A bimatrix A_B is defined as the union of two rectangular array of numbers A_1 and A_2 arranged into rows and columns. It is written as $A_B = A_1 \cup A_2$ with $A_1 \neq A_2$ (except zero and unit bimatrices) where,

$$A_1 = \begin{bmatrix} a_{11}^1 & a_{12}^1 & \cdots & a_{1n}^1 \\ a_{21}^1 & a_{22}^1 & \cdots & a_{2n}^1 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}^1 & a_{m2}^1 & \cdots & a_{mn}^1 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} a_{11}^2 & a_{12}^2 & \cdots & a_{1n}^2 \\ a_{21}^2 & a_{22}^2 & \cdots & a_{2n}^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}^2 & a_{m2}^2 & \cdots & a_{mn}^2 \end{bmatrix}$$

' \cup ' is just for the notational convenience (symbol) only.

Definition 2.2 [5]

Let $A_B = A_1 \cup A_2$ and $C_B = C_1 \cup C_2$ be any two $m \times n$ bimatrices. The sum D_B of the bimatrices A_B and C_B is defined as $D_B = A_B + C_B = (A_1 \cup A_2) + (C_1 \cup C_2) = (A_1 + C_1) \cup (A_2 + C_2)$. Where $A_1 + C_1$ and $A_2 + C_2$ are the usual addition of matrices.

Definition 2.3 [6]

If $A_B = A_1 \cup A_2$ and $C_B = C_1 \cup C_2$ be two bimatrices, then A_B and C_B are said to be equal (written as $A_B = C_B$) if and only if A_1 and C_1 are identical and A_2 and C_2 are identical. (That is, $A_1 = C_1$ and $A_2 = C_2$).

Definition 2.4 [6]

Given a bimatrix $A_B = A_1 \cup A_2$ and a scalar λ , the product of λ and A_B written as λA_B is defined to be

$$\lambda A_B = \left[\begin{array}{cccc} \lambda a_{11}^1 & \lambda a_{12}^1 & \cdots & \lambda a_{1n}^1 \\ \lambda a_{21}^1 & \lambda a_{22}^1 & \cdots & \lambda a_{2n}^1 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1}^1 & \lambda a_{m2}^1 & \cdots & \lambda a_{mn}^1 \end{array} \right] \cup \left[\begin{array}{cccc} \lambda a_{11}^2 & \lambda a_{12}^2 & \cdots & \lambda a_{1n}^2 \\ \lambda a_{21}^2 & \lambda a_{22}^2 & \cdots & \lambda a_{2n}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1}^2 & \lambda a_{m2}^2 & \cdots & \lambda a_{mn}^2 \end{array} \right]$$

$$= (\lambda A_1 \cup \lambda A_2).$$

That is, each element of A_1 and A_2 are multiplied by λ .

Remark 2.5 [6]

If $A_B = A_1 \cup A_2$ be a bimatrix, then we call A_1 and A_2 as the component matrices of the bimatrix A_B .

Definition 2.6 [5]

If $A_B = A_1 \cup A_2$ and $C_B = C_1 \cup C_2$ are both $n \times n$ square bimatrices then, the bimatrix multiplication is defined as, $A_B \times C_B = (A_1 C_1) \cup (A_2 C_2)$.

Definition 2.7 [5]

Let $A_B^{m \times m} = A_1 \cup A_2$ be a $m \times m$ square bimatrix. We define $I_B^{m \times m} = I^{m \times m} \cup I^{m \times m} = I_1^{m \times m} \cup I_2^{m \times m}$ to be the identity bimatrix.

Definition 2.8 [5]

Let $A_B^{m \times m} = A_1 \cup A_2$ be a square bimatrix, A_B is a symmetric bimatrix if the component matrices A_1 and A_2 are symmetric matrices. i.e., $A_1 = A_1^T$ and $A_2 = A_2^T$.

Definition 2.9 [5]

Let $A_B^{m \times m} = A_1 \cup A_2$ be a $m \times m$ square bimatrix i.e., A_1 and A_2 are $m \times m$ square matrices. A skew-symmetric bimatrix is a bimatrix A_B for which $A_B = -A_B^T$, where $-A_B^T = -A_1^T \cup -A_2^T$ i.e., the component matrices A_1 and A_2 are skew-symmetric.

Definition 2.10 [4]

A bimatrix $A_B = A_1 \cup A_2$ is said to be orthogonal bimatrix, if $A_B A_B^T = A_B^T A_B = I_B$ (i.e., the component matrices of A_B are orthogonal). That is, $A_B^T = A_B^{-1}$.

Definition 2.11

Let $A_B^{m \times m} = A_1 \cup A_2$ be a square bimatrix, A_B is a secondary symmetric bimatrix if the component matrices A_1 and A_2 are secondary symmetric matrices. i.e., $A_1 = A_1^S = V_1 A_1^T V_1$ and $A_2 = A_2^S = V_2 A_2^T V_2$ where V_1 and V_2 are the component matrices of a permutation bimatrix V_B with units in its secondary diagonal.

Definition 2.12

Let $A_B^{m \times m} = A_1 \cup A_2$ be a $m \times m$ square bimatrix i.e., A_1 and A_2 are $m \times m$ square matrices. A secondary skew-symmetric bimatrix is a bimatrix A_B for which $A_B = -A_B^S$, where $-A_B^S = -A_1^S \cup -A_2^S$ i.e., the component matrices A_1 and A_2 are secondary skew-symmetric.

III. SECONDARY ORTHOGONAL BIMATRICES**Definition 3.1**

A bimatrix $A_B = A_1 \cup A_2$ is said to be secondary orthogonal bimatrix, if $A_B V_B A_B^T V_B = V_B A_B^T V_B A_B = I_B$ (or) $A_B A_B^S = A_B^S A_B = I_B$ where V_B is a permutation bimatrix with units in its secondary diagonal. (That is, the component matrices of A_B are secondary orthogonal.)

That is, $A_B^S = A_B^{-1}$ (or) $(A_1^S \cup A_2^S) = (A_1^{-1} \cup A_2^{-1})$.

Remark 3.2

Let $A_B = A_1 \cup A_2$ be a secondary orthogonal bimatrix. If A_1 and A_2 are square and possess the same order then A_B is called square secondary orthogonal bimatrix, and if A_1 and A_2 are of different orders then A_B is called mixed square secondary orthogonal bimatrix.

Theorem 3.3

Let $A_B = A_1 \cup A_2$ and $B_B = B_1 \cup B_2$ be given bimatrices.

- (1) If A_B is secondary symmetric bimatrix then A_B^T and λA_B are secondary symmetric bimatrices for an arbitrary λ .
- (2) If A_B and B_B are secondary symmetric bimatrices of the same order then $A_B \pm B_B$ are secondary symmetric bimatrices as well.
- (3) If A_B is secondary symmetric bimatrix then its Moore-Penrose inverse A_B^\dagger is secondary symmetric bimatrix.

Proof

$A_B = A_1 \cup A_2$ is secondary symmetric bimatrix, if

$$A_B^S = A_B \text{ (i.e., } (A_1 \cup A_2)^S = V_1 A_1^T V_1 \cup V_2 A_2^T V_2 = A_1 \cup A_2 \text{)}.$$

$$(1) (A_B^T)^S = (A_B^S)^T = (A_1^S \cup A_2^S)^T = (A_1 \cup A_2)^T \text{ and}$$

$$(\lambda A_B)^S = (\lambda A_1 \cup \lambda A_2)^S = \lambda A_1^S \cup \lambda A_2^S = \lambda A_1 \cup \lambda A_2 = \lambda (A_1 \cup A_2) = \lambda A_B.$$

$$(2) (A_B \pm B_B)^S = [(A_1 \cup A_2) \pm (B_1 \cup B_2)]^S$$

$$= (A_1 \cup A_2)^S \pm (B_1 \cup B_2)^S$$

$$= (A_1 \cup A_2) \pm (B_1 \cup B_2)$$

$$= A_B \pm B_B.$$

$$(3) \quad (A_B^\dagger)^S = (A_1^\dagger \cup A_2^\dagger)^S = (A_1^S \cup A_2^S)^\dagger = (A_1 \cup A_2)^\dagger = A_B^\dagger.$$

Theorem 3.4

If $A_B = A_1 \cup A_2$ is secondary symmetric bimatrix, then $A_B^S A_B$ and $A_B + A_B^S$ are secondary symmetric bimatrices.

Proof

Let $A_B = A_1 \cup A_2$ is secondary symmetric bimatrix, if $A_B^S = A_B$

$$A_B^S = A_B \quad (i.e., (A_1 \cup A_2)^S = V_1 A_1^T V_1 \cup V_2 A_2^T V_2 = A_1 \cup A_2).$$

$$\begin{aligned} \text{Hence, } (A_B^S A_B)^S &= [(A_1^S \cup A_2^S)(A_1 \cup A_2)]^S \\ &= (A_1^S A_1 \cup A_2^S A_2)^S \\ &= (A_1^S A_1) \cup (A_2^S A_2) \\ &= (A_1^S \cup A_2^S)(A_1 \cup A_2) \\ &= A_B^S A_B. \end{aligned}$$

$$\begin{aligned} \text{And } (A_B + A_B^S)^S &= [(A_1 \cup A_2) + (A_1^S \cup A_2^S)]^S \\ &= [(A_1 + A_1^S) \cup (A_2 + A_2^S)]^S \\ &= (A_1^S + A_1) \cup (A_2^S + A_2) \\ &= (A_1 \cup A_2) + (A_1^S \cup A_2^S) \\ &= A_B + A_B^S. \end{aligned}$$

Theorem 3.5

- (1) If $A_B = A_1 \cup A_2$ is secondary skew-symmetric bimatrix then $A_B - A_B^S$ is secondary skew-symmetric bimatrix.
- (2) If A_B and B_B are secondary skew symmetric bimatrices of the same order then $A_B \pm B_B$ are secondary skew symmetric bimatrices as well.

Proof

- (1) Let $A_B = A_1 \cup A_2$ is secondary skew-symmetric bimatrix, if $A_B^S = -A_B$.

$$(i.e., (A_1 \cup A_2)^S = A_1^S \cup A_2^S = V_1 A_1^T V_1 \cup V_2 A_2^T V_2 = -A_1 \cup -A_2).$$

$$\begin{aligned} \text{Hence, } (A_B - A_B^S)^S &= [(A_1 \cup A_2) - (V_1 A_1^T V_1 \cup V_2 A_2^T V_2)]^S \\ &= [(A_1 - V_1 A_1^T V_1) \cup (A_2 - V_2 A_2^T V_2)]^S \\ &= (A_1 - V_1 A_1^T V_1)^S \cup (A_2 - V_2 A_2^T V_2)^S \\ &= (V_1 A_1^T V_1 - A_1) \cup (V_2 A_2^T V_2 - A_2) \end{aligned}$$

$$\begin{aligned}
 &= -(A_1 - V_1 A_1^T V_1) \cup -(A_2 - V_2 A_2^T V_2) \\
 &= -[(A_1 \cup A_2) - (V_1 A_1^T V_1 \cup V_2 A_2^T V_2)] \\
 &= -[(A_1 \cup A_2) - (A_1^S \cup A_2^S)] \\
 &= -(A_B - A_B^S).
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad (A_B \pm B_B)^S &= [(A_1 \cup A_2) \pm (B_1 \cup B_2)]^S \\
 &= (A_1 \cup A_2)^S \pm (B_1 \cup B_2)^S \\
 &= (V_1 A_1^T V_1 \cup V_2 A_2^T V_2) \pm (V_1 B_1^T V_1 \cup V_2 B_2^T V_2) \\
 &= [- (A_1 \cup A_2)] \pm [- (B_1 \cup B_2)] \\
 &= -[(A_1 \cup A_2) \pm (B_1 \cup B_2)] \\
 &= -(A_B \pm B_B).
 \end{aligned}$$

Theorem 3.6

Let $A_B = A_1 \cup A_2$ and $B_B = B_1 \cup B_2$ be given bimatrices.

- (1) If A_B is secondary skew symmetric bimatrix then A_B^T is secondary skew symmetric bimatrix as well.
- (2) If A_B is secondary skew symmetric bimatrix then λA_B is secondary skew symmetric bimatrix for an arbitrary λ .
- (3) If A_B is secondary skew symmetric bimatrix then its Moore-Penrose inverse A_B^\dagger is secondary skew symmetric bimatrix.

Proof

$A_B = A_1 \cup A_2$ is secondary skew symmetric bimatrix, if $A_B^S = (V_1 A_1^T V_1 \cup V_2 A_2^T V_2) = -(A_1 \cup A_2)$

- (1) $(A_B^T)^S = (A_B^S)^T = (A_1^S \cup A_2^S)^T = (A_1^S)^T \cup (A_2^S)^T = (A_1^T \cup A_2^T) = -A_B^T$.
- (2) $(\lambda A_B)^S = (\lambda A_1 \cup \lambda A_2)^S = \lambda V_1 A_1^T V_1 \cup \lambda V_2 A_2^T V_2 = \lambda (V_1 A_1^T V_1 \cup V_2 A_2^T V_2) = -\lambda (A_1 \cup A_2) = -\lambda A_B$.
- (3) $(A_B^\dagger)^S = (A_1^\dagger \cup A_2^\dagger)^S = (A_1^S \cup A_2^S)^\dagger = -(A_1 \cup A_2)^\dagger = -A_B^\dagger$.

Corollary 3.7

Every square bimatrix can be written as a sum of secondary symmetric and secondary skew symmetric bimatrices.

Proof

By Theorem 3.4 and 3.5 we have, $(A_B + A_B^S)$ is secondary symmetric bimatrix and $(A_B - A_B^S)$ is secondary skew symmetric bimatrix.

$$\begin{aligned}
 \text{Therefore } A_B &= (A_1 \cup A_2) = \frac{1}{2}(A_B + A_B^S) + \frac{1}{2}(A_B - A_B^S) \\
 &= \frac{1}{2}[(A_1 \cup A_2) + (A_1^S \cup A_2^S)] + \frac{1}{2}[(A_1 \cup A_2) - (A_1^S \cup A_2^S)] \\
 &= \frac{1}{2}[(A_1 + A_1^S) \cup (A_2 + A_2^S)] + \frac{1}{2}[(A_1 - A_1^S) \cup (A_2 - A_2^S)].
 \end{aligned}$$

Definition 3.8

Let $A_B = (A_1 \cup A_2) \in C_{n \times n}$ and $B_B = (B_1 \cup B_2) \in C_{n \times n}$ be given bimatrices. A_B is said to be secondary orthogonally bisimilar to B_B , there exists a secondary orthogonal bimatrix $Q_B = (Q_1 \cup Q_2) \in C_{n \times n}$ such that $A_B = Q_B^S B_B Q_B$.

$$(i.e., (A_1 \cup A_2) = (Q_1^S \cup Q_2^S)(B_1 \cup B_2)(Q_1 \cup Q_2) = (Q_1^S B_1 Q_1) \cup (Q_2^S B_2 Q_2)).$$

That is, the components A_1 and A_2 are secondary orthogonally similar to B_1 and B_2 .

Remark 3.9

Secondary orthogonal bimatrices also characterized by their inverses. Before discussing this, we need a brief digression on the secondary transpose of a bimatrix A_B . The main properties of secondary transpose which we have seen are $(A_B^S)^S = A_B$, $(A_B B_B)^S = B_B^S A_B^S$, $\det(A_B B_B) = \det(A_B) = \det(B_B)$, $\det(A_B) = \det(A_B^S)$, and $\det(I_B) = 1$ (i.e., determinant value of the identity components I_1 and I_2 are 1).

Theorem 3.10

- (1) If $A_B = A_1 \cup A_2$ is secondary orthogonal bimatrix then $\det A_B = \pm 1$
- (2) If $A_B = A_1 \cup A_2$ is secondary orthogonal bimatrix then $-A_B, A_B^S$ and A_B^{-1} are secondary orthogonal bimatrices as well
- (3) If $A_B = A_1 \cup A_2$ and $B_B = B_1 \cup B_2$ are secondary orthogonal bimatrices then $A_B B_B$ is also secondary orthogonal bimatrix.

Proof

$A_B = A_1 \cup A_2$ is secondary orthogonal bimatrix, if $A_B^S = A_B^{-1}$ (i.e., $(A_1^S \cup A_2^S) = (A_1^{-1} \cup A_2^{-1})$).

$$\begin{aligned} (1) \quad 1 &= \det(I_B) = \det(A_B A_B^S) \\ &= \det[(A_1 \cup A_2)(V_1 A_1^T V_1 \cup V_2 A_2^T V_2)] = \det(A_1 \cup A_2) \det(V_1 A_1^T V_1 \cup V_2 A_2^T V_2) \\ &= \det(A_1 \cup A_2) \det(A_1 \cup A_2) = [\det(A_1 \cup A_2)]^2. \end{aligned}$$

$$\begin{aligned} (2) \quad (-A_B)^S &= [-(A_1 \cup A_2)]^S \\ &= [- (A_1^S \cup A_2^S)] \\ &= [-(A_1^{-1} \cup A_2^{-1})] \\ &= [-(A_1 \cup A_2)]^{-1} \\ &= (-A_B)^{-1}, \end{aligned}$$

$$(A_B^T)^S = (A_B^S)^T = (A_1^S \cup A_2^S)^T = (A_1^{-1} \cup A_2^{-1})^T = (A_B^{-1})^T = (A_B^T)^{-1}.$$

and

$$(A_B^{-1})^S = (A_1^{-1} \cup A_2^{-1})^S = (V_1 A_1^T V_1 \cup V_2 A_2^T V_2)^{-1} = (A_1^{-1} \cup A_2^{-1})^{-1} = (A_B^{-1})^{-1}.$$

$$\begin{aligned} (3) \quad (A_B B_B)^S &= [(A_1 \cup A_2)(B_1 \cup B_2)]^S = (B_1 \cup B_2)^S (A_1 \cup A_2)^S \\ &= (B_1^S \cup B_2^S)(A_1^S \cup A_2^S) \\ &= (B_1^{-1} \cup B_2^{-1})(A_1^{-1} \cup A_2^{-1}) \end{aligned}$$

$$\begin{aligned}
 &= (B_1 \cup B_2)^{-1} (A_1 \cup A_2)^{-1} \\
 &= \left[(A_1 \cup A_2) (B_1 \cup B_2) \right]^{-1} \\
 &= (A_B B_B)^{-1}.
 \end{aligned}$$

Theorem 3.11

If $A_B = A_1 \cup A_2$ is real secondary skew symmetric bimatrix such that $A_B^2 + I_B = O_B$ (i.e., $(A_1 \cup A_2)^2 + (I_1 \cup I_2) = O_B$) then A_B is secondary orthogonal bimatrix and is of even order.

Proof

$$\begin{aligned}
 &A_B = A_1 \cup A_2 \text{ is real secondary skew symmetric bimatrix, if } A_B^S = -A_B \\
 \Rightarrow &(A_1^S \cup A_2^S) = -(A_1 \cup A_2) \\
 \Rightarrow &(A_1 \cup A_2) = -(A_1^S \cup A_2^S) \\
 \Rightarrow &(A_1 \cup A_2)(A_1 \cup A_2) = -(A_1 \cup A_2)(A_1^S \cup A_2^S) \\
 \Rightarrow &(A_1 \cup A_2)^2 = -(A_1 \cup A_2)(A_1^S \cup A_2^S) \\
 \Rightarrow &O_B - (I_1 \cup I_2) = -(A_1 \cup A_2)(A_1^S \cup A_2^S) \\
 \Rightarrow &(I_1 \cup I_2) = (A_1 \cup A_2)(A_1^S \cup A_2^S) \\
 \Rightarrow &I_B = A_B + A_B^S.
 \end{aligned}$$

Again $|A_1 \cup A_2| = |A_1^S \cup A_2^S|$ and $|kA_1 \cup kA_2|^n = k^n |A_1 \cup A_2|$, where n is the order of A_B .

Since, $A_B^S = (-1)A_B$

$$\begin{aligned}
 \Rightarrow &(V_1 A_1^T V_1 \cup V_2 A_2^T V_2) = (-1)(A_1 \cup A_2) \\
 \Rightarrow &|V_1 A_1^T V_1 \cup V_2 A_2^T V_2| = (-1)^n |A_1 \cup A_2| \\
 \Rightarrow &[1 - (-1)^n] |A_1 \cup A_2| = 0.
 \end{aligned}$$

Hence, either $|A_1 \cup A_2| = 0$ or $1 - (-1)^n = 0$, that is n is even.

$$\begin{aligned}
 \text{But } &(A_1 \cup A_2)^2 = O_B - (I_1 \cup I_2) = -(I_1 \cup I_2) \\
 \Rightarrow &|A_1 \cup A_2|^2 = (-1)^n |I_1 \cup I_2| = (-1)^n \neq 0.
 \end{aligned}$$

Hence the only possibility left that A_B is of even order.

Theorem 3.12

- (1) If $A_B = A_1 \cup A_2$ is secondary symmetric bimatrix such that $A_B^2 = I_B$ then A_B is secondary orthogonal bimatrix
- (2) If $A_B = A_1 \cup A_2$ is a square bimatrix and $A_B - \frac{1}{2}I_B$ and $A_B + \frac{1}{2}I_B$ are secondary orthogonal bimatrices then A_B is secondary skew symmetric bimatrix.

Proof

(1) $A_B = A_1 \cup A_2$ is secondary symmetric bimatrix and $(A_1 \cup A_2)^2 = I_1 \cup I_2$.

Hence, $I_1 \cup I_2 = (A_1 \cup A_2)^2 = (A_1 \cup A_2)(A_1 \cup A_2) = (A_1 \cup A_2)(A_1^S \cup A_2^S)$.

Similarly we can prove $(A_1 \cup A_2)(A_1^S \cup A_2^S) = I_1 \cup I_2$.

Therefore $(A_1 \cup A_2)(A_1^S \cup A_2^S) = (A_1^S \cup A_2^S)(A_1 \cup A_2) = I_1 \cup I_2$.

Hence, $A_B A_B^S = A_B V_B A_B^T V_B = I_B$.

(2) Since $\left(A_B + \frac{1}{2}I_B\right)\left(A_B + \frac{1}{2}I_B\right)^S = I_B$ and $\left(A_B - \frac{1}{2}I_B\right)\left(A_B - \frac{1}{2}I_B\right)^S = I_B$

$$\Rightarrow \left[(A_1 \cup A_2) + \frac{1}{2}(I_1 \cup I_2)\right] \left[(A_1 \cup A_2) + \frac{1}{2}(I_1 \cup I_2)\right]^S \\ = \left[(A_1 \cup A_2) - \frac{1}{2}(I_1 \cup I_2)\right] \left[(A_1 \cup A_2) - \frac{1}{2}(I_1 \cup I_2)\right]^S$$

$$\Rightarrow \left[(A_1 \cup A_2) + (A_1^S \cup A_2^S)\right] = -\left[(A_1 \cup A_2) - (A_1^S \cup A_2^S)\right]$$

$$\Rightarrow (A_1^S \cup A_2^S) = -(A_1 \cup A_2).$$

$$\Rightarrow A_B^S = V_B A_B^T V_B = -A_B.$$

Theorem 3.13

Let $J_B = J_1 \cup J_2$ be the diagonal bimatrix with entries ± 1 and $A_B = A_1 \cup A_2$ is secondary orthogonal bimatrix then

(a) $J_B A_B = (J_1 A_1 \cup J_2 A_2)$ is secondary orthogonal bimatrix.

(b) $A_B J_B = (A_1 J_1 \cup A_2 J_2)$ is secondary orthogonal bimatrix.

(c) $A_B J_B A_B$ is secondary orthogonal bimatrix.

(d) $J_B A_B J_B$ is secondary orthogonal bimatrix.

Proof

$A_B = A_1 \cup A_2$ is secondary orthogonal bimatrix, if $A_B^S = A_B^{-1}$ (i.e., $(A_1^S \cup A_2^S) = (A_1^{-1} \cup A_2^{-1})$).

Hence

$$\begin{aligned} \text{(a)} \quad (J_B A_B)^S &= \left[(J_1 \cup J_2)(A_1 \cup A_2)\right]^S \\ &= (A_1 \cup A_2)^S (J_1 \cup J_2)^S \\ &= (A_1^{-1} \cup A_2^{-1})(J_1^{-1} \cup J_2^{-1}) \\ &= (A_1 \cup A_2)^{-1} (J_1 \cup J_2)^{-1} \\ &= \left[(J_1 \cup J_2)(A_1 \cup A_2)\right]^{-1} \\ &= (J_B A_B)^{-1}. \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad (A_B J_B)^S &= [(A_1 \cup A_2)(J_1 \cup J_2)]^S \\
 &= (J_1 \cup J_2)^S (A_1 \cup A_2)^S \\
 &= (J_1^{-1} \cup J_2^{-1})(A_1^{-1} \cup A_2^{-1}) \\
 &= (J_1 \cup J_2)^{-1} (A_1 \cup A_2)^{-1} \\
 &= [(A_1 \cup A_2)(J_1 \cup J_2)]^{-1} \\
 &= (A_B J_B)^{-1}.
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad (A_B J_B A_B)^S &= [(A_1 \cup A_2)(J_1 \cup J_2)(A_1 \cup A_2)]^S \\
 &= (A_1 \cup A_2)^S [(A_1 \cup A_2)(J_1 \cup J_2)]^S \\
 &= (A_1 \cup A_2)^{-1} [(A_1 \cup A_2)(J_1 \cup J_2)]^{-1} \\
 &= [(A_1 \cup A_2)(J_1 \cup J_2)(A_1 \cup A_2)]^{-1} \\
 &= (A_B J_B A_B)^{-1}.
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad (J_B A_B J_B)^S &= [(J_1 \cup J_2)(A_1 \cup A_2)(J_1 \cup J_2)]^S \\
 &= (J_1 \cup J_2)^S [(J_1 \cup J_2)(A_1 \cup A_2)]^S \\
 &= (J_1 \cup J_2)^{-1} [(J_1 \cup J_2)(A_1 \cup A_2)]^{-1} \\
 &= [(J_1 \cup J_2)(A_1 \cup A_2)(J_1 \cup J_2)]^{-1} \\
 &= (J_B A_B J_B)^{-1}.
 \end{aligned}$$

Theorem 3.14

If $A_B = A_1 \cup A_2$ is secondary orthogonal bimatrix such that $(A_1^S \cup A_2^S) = (V_1 A_1^T V_1 \cup V_2 A_2^T V_2) = (A_1 \cup A_2)$ and $(A_1^2 \cup A_2^2) = (I_1 \cup I_2)$ then $A_{B1} + A_{B2} = I_B$ and $A_{B1} A_{B2} = O_B$, where $A_{B1} = \frac{1}{2}[(I_1 \cup I_2) + (A_1 \cup A_2)]$ and $A_{B2} = \frac{1}{2}[(I_1 \cup I_2) - (A_1 \cup A_2)]$ are projection bimatrices.

Proof

$$\begin{aligned}
 A_B &= A_1 \cup A_2 \text{ is secondary orthogonal bimatrix, } (A_1^S \cup A_2^S) = (V_1 A_1^T V_1 \cup V_2 A_2^T V_2) = (A_1 \cup A_2) \text{ and } \\
 (A_1^2 \cup A_2^2) &= (I_1 \cup I_2) \\
 \Rightarrow A_{B1} + A_{B2} &= \frac{1}{2}[(I_1 \cup I_2) + (A_1 \cup A_2)] + \frac{1}{2}[(I_1 \cup I_2) - (A_1 \cup A_2)] \\
 \Rightarrow A_{B1} + A_{B2} &= I_B. \quad \text{And} \\
 \Rightarrow A_{B1} A_{B2} &= \frac{1}{4}[(I_1 \cup I_2) + (A_1 \cup A_2)][(I_1 \cup I_2) - (A_1 \cup A_2)]
 \end{aligned}$$

$$= \frac{1}{4} \left[(I_1 \cup I_2)^2 - (A_1 \cup A_2)^2 \right]$$

$$= \frac{1}{4} \left[(I_1 \cup I_2) - (I_1 \cup I_2) \right]$$

$$A_{B1} A_{B2} = O_B.$$

Theorem 3.15

Let $I_B = (I_1 \cup I_2)$ be a given identity bimatrix and let $Q_B = (Q_1 \cup Q_2) \in C_{n \times n}$ be a secondary orthogonal bimatrix, and set $R_B = (R_1 \cup R_2) \equiv Q_B I_B V_B Q_B^T V_B \equiv Q_B I_B Q_B^S$ and $S_B = (S_1 \cup S_2) \equiv Q_B A_B V_B Q_B^T V_B \equiv Q_B A_B Q_B^S$. Then

- (a) $A_B = A_1 \cup A_2 \in C_{n \times n}$ is secondary orthogonal bimatrix if and only if $S_B^S R_B S_B = R_B$.
- (b) $A_B = A_1 \cup A_2 \in C_{n \times n}$ is secondary symmetric bimatrix if and only if $S_B^S R_B = R_B S_B$.
- (c) $A_B = A_1 \cup A_2 \in C_{n \times n}$ is secondary skew symmetric bimatrix if and only if $S_B^S R_B = -R_B S_B$.

Proof

(a) $S_B^S R_B S_B = R_B$

$$\Leftrightarrow (S_1^S \cup S_2^S)(R_1 \cup R_2)(S_1 \cup S_2) = (R_1 \cup R_2) \Leftrightarrow [(Q_1 \cup Q_2)(A_1 \cup A_2)(Q_1^S \cup Q_2^S)]^S$$

$$[(Q_1 \cup Q_2)(I_1 \cup I_2)(Q_1^S \cup Q_2^S)] [(Q_1 \cup Q_2)(A_1 \cup A_2)(Q_1^S \cup Q_2^S)]$$

$$= [(Q_1 \cup Q_2)(I_1 \cup I_2)(Q_1^S \cup Q_2^S)] \Leftrightarrow [(Q_1 \cup Q_2)(A_1^S \cup A_2^S)(Q_1^S \cup Q_2^S)]$$

$$[(Q_1 \cup Q_2)(I_1 \cup I_2)(Q_1^S \cup Q_2^S)] [(Q_1 \cup Q_2)(A_1 \cup A_2)(Q_1^S \cup Q_2^S)]$$

$$= [(Q_1 \cup Q_2)(I_1 \cup I_2)(Q_1^S \cup Q_2^S)] \Leftrightarrow [Q_1 A_1^S Q_1^S \cup Q_2 A_2^S Q_2^S] [Q_1 I_1 Q_1^S \cup Q_2 I_2 Q_2^S]$$

$$[Q_1 A_1 Q_1^S \cup Q_2 A_2 Q_2^S] = [Q_1 I_1 Q_1^S \cup Q_2 I_2 Q_2^S]$$

$$\Leftrightarrow [(Q_1 A_1^S Q_1^S Q_1 I_1 Q_1^S Q_1 A_1 Q_1^S) \cup (Q_2 A_2^S Q_2^S Q_2 I_2 Q_2^S Q_2 A_2 Q_2^S)]$$

$$= [(Q_1 I_1 Q_1^S) \cup (Q_2 I_2 Q_2^S)]$$

$$\Leftrightarrow [(Q_1 A_1^S I_1 A_1 Q_1^S) \cup (Q_2 A_2^S I_2 A_2 Q_2^S)] = [(Q_1 I_1 Q_1^S) \cup (Q_2 I_2 Q_2^S)]$$

$$\Leftrightarrow [(Q_1 A_1^S A_1 Q_1^S) \cup (Q_2 A_2^S A_2 Q_2^S)] = [(Q_1 I_1 Q_1^S) \cup (Q_2 I_2 Q_2^S)]$$

$$\Leftrightarrow (Q_1 \cup Q_2)(A_1^S A_1 \cup A_2^S A_2)(Q_1^S \cup Q_2^S) = (Q_1 \cup Q_2)(I_1 \cup I_2)(Q_1^S \cup Q_2^S)$$

$$\Leftrightarrow (A_1^S A_1 \cup A_2^S A_2) = (I_1 \cup I_2)$$

$$\Leftrightarrow (A_1^S \cup A_2^S)(A_1 \cup A_2) = (I_1 \cup I_2)$$

$$\Leftrightarrow A_B^S A_B = I_B.$$

(b) $S_B^S R_B = R_B S_B \Leftrightarrow (S_1^S \cup S_2^S)(R_1 \cup R_2) = (R_1 \cup R_2)(S_1 \cup S_2)$

$$\Leftrightarrow [(Q_1 \cup Q_2)(A_1 \cup A_2)(Q_1^S \cup Q_2^S)]^S [(Q_1 \cup Q_2)(I_1 \cup I_2)(Q_1^S \cup Q_2^S)]$$

$$\begin{aligned}
&= [(Q_1 \cup Q_2)(I_1 \cup I_2)(Q_1^S \cup Q_2^S)] [(Q_1 \cup Q_2)(A_1 \cup A_2)(Q_1^S \cup Q_2^S)] \\
&\Leftrightarrow [(Q_1 \cup Q_2)(A_1^S \cup A_2^S)(Q_1^S \cup Q_2^S)] [(Q_1 \cup Q_2)(I_1 \cup I_2)(Q_1^S \cup Q_2^S)] \\
&= [(Q_1 \cup Q_2)(I_1 \cup I_2)(Q_1^S \cup Q_2^S)] [(Q_1 \cup Q_2)(A_1 \cup A_2)(Q_1^S \cup Q_2^S)] \\
&\Leftrightarrow [(Q_1 A_1^S Q_1^S) \cup (Q_2 A_2^S Q_2^S)] [(Q_1 I_1 Q_1^S) \cup (Q_2 I_2 Q_2^S)] \\
&= [(Q_1 I_1 Q_1^S \cup Q_2 I_2 Q_2^S)] \cup [(Q_1 A_1 Q_1^S) \cup (Q_2 A_2 Q_2^S)] \\
&\Leftrightarrow [(Q_1 A_1^S Q_1^S Q_1 I_1 Q_1^S) \cup (Q_2 A_2^S Q_2^S Q_2 I_2 Q_2^S)] \\
&= [(Q_1 I_1 Q_1^S Q_1 A_1 Q_1^S)] \cup [(Q_2 I_2 Q_2^S Q_2 A_2 Q_2^S)] \\
&\Leftrightarrow [(Q_1 A_1^S I_1 Q_1^S) \cup (Q_2 A_2^S I_2 Q_2^S)] = [(Q_1 I_1 A_1 Q_1^S)] \cup [(Q_2 I_2 A_2 Q_2^S)] \\
&\Leftrightarrow [(Q_1 A_1^S Q_1^S) \cup (Q_2 A_2^S Q_2^S)] = [(Q_1 A_1 Q_1^S) \cup (Q_2 A_2 Q_2^S)] \\
&\Leftrightarrow (Q_1 \cup Q_2)(A_1^S \cup A_2^S)(Q_1^S \cup Q_2^S) = (Q_1 \cup Q_2)(A_1 \cup A_2)(Q_1^S \cup Q_2^S) \\
&\Leftrightarrow (A_1^S \cup A_2^S) = (A_1 \cup A_2) \quad \Leftrightarrow A_B^S = A_B.
\end{aligned}$$

(c) $S_B^S R_B = -R_B S_B$

$$\begin{aligned}
&\Leftrightarrow (S_1^S \cup S_2^S)(R_1 \cup R_2) = -(R_1 \cup R_2)(S_1 \cup S_2) \\
&\Leftrightarrow [(Q_1 \cup Q_2)(A_1 \cup A_2)(Q_1^S \cup Q_2^S)]^S [(Q_1 \cup Q_2)(I_1 \cup I_2)(Q_1^S \cup Q_2^S)] \\
&= -[(Q_1 \cup Q_2)(I_1 \cup I_2)(Q_1^S \cup Q_2^S)] [(Q_1 \cup Q_2)(A_1 \cup A_2)(Q_1^S \cup Q_2^S)] \\
&\Leftrightarrow [(Q_1 \cup Q_2)(A_1^S \cup A_2^S)(Q_1^S \cup Q_2^S)] [(Q_1 \cup Q_2)(I_1 \cup I_2)(Q_1^S \cup Q_2^S)] \\
&= -[(Q_1 \cup Q_2)(I_1 \cup I_2)(Q_1^S \cup Q_2^S)] [(Q_1 \cup Q_2)(A_1 \cup A_2)(Q_1^S \cup Q_2^S)] \\
&\Leftrightarrow [(Q_1 A_1^S Q_1^S) \cup (Q_2 A_2^S Q_2^S)] [(Q_1 I_1 Q_1^S) \cup (Q_2 I_2 Q_2^S)] \\
&= -[(Q_1 I_1 Q_1^S \cup Q_2 I_2 Q_2^S)] \cup [(Q_1 A_1 Q_1^S) \cup (Q_2 A_2 Q_2^S)] \\
&\Leftrightarrow [(Q_1 A_1^S Q_1^S Q_1 I_1 Q_1^S) \cup (Q_2 A_2^S Q_2^S Q_2 I_2 Q_2^S)] \\
&= -[(Q_1 I_1 Q_1^S Q_1 A_1 Q_1^S)] \cup [(Q_2 I_2 Q_2^S Q_2 A_2 Q_2^S)] \\
&\Leftrightarrow [(Q_1 A_1^S I_1 Q_1^S) \cup (Q_2 A_2^S I_2 Q_2^S)] = -[(Q_1 I_1 A_1 Q_1^S)] \cup [(Q_2 I_2 A_2 Q_2^S)] \\
&\Leftrightarrow [(Q_1 A_1^S Q_1^S) \cup (Q_2 A_2^S Q_2^S)] = -[(Q_1 A_1 Q_1^S) \cup (Q_2 A_2 Q_2^S)] \\
&\Leftrightarrow [(Q_1 \cup Q_2)(A_1^S \cup A_2^S)(Q_1^S \cup Q_2^S)] = -[(Q_1 \cup Q_2)(A_1 \cup A_2)(Q_1^S \cup Q_2^S)] \\
&\Leftrightarrow (A_1^S \cup A_2^S) = -(A_1 \cup A_2) \quad \Leftrightarrow A_B^S = -A_B.
\end{aligned}$$

Theorem 3.16

- (a) If $A_B = A_1 \cup A_2$ is non-singular bimatrix, then $A_B = G_B Q_B$, where $Q_B = (Q_1 \cup Q_2) \in C_{n \times n}$ is complex secondary orthogonal bimatrix, $G_B = (G_1 \cup G_2) \in C_{n \times n}$ is complex secondary symmetric bimatrix and G_B is a bipolynomial in $A_B A_B^S$.
- (b) Suppose $A_B = G_B Q_B$, where $G_B = G_B^S$ and $Q_B Q_B^S = I_B$. Then $G_B^2 = A_B A_B^S$. If G_B commute with Q_B , then A_B commutes with A_B^S . Conversely, if A_B commutes A_B^S and G_B is a bipolynomial in $A_B A_B^S$, then G_B commutes with Q_B .

Proof

- (a) If $A_B = G_B Q_B$ for some secondary symmetric bimatrix G_B (i.e., $(V_1 G_1^T V_1 \cup V_2 G_2^T V_2) = (G_1 \cup G_2)$) and secondary orthogonal bimatrix Q_B , then

$$\begin{aligned} A_B V_B A_B^T V_B &= G_B Q_B V_B Q_B^T G_B^T V_B \\ &= (G_1 \cup G_2) (Q_1 \cup Q_2) (V_1 \cup V_2) (Q_1^T \cup Q_2^T) (G_1^T \cup G_2^T) (V_1 \cup V_2) \\ &= (G_1 Q_1 V_1 Q_1^T G_1^T V_1 \cup G_2 Q_2 V_2 Q_2^T G_2^T V_2) \\ &= (G_1 V_1 G_1^T V_1 \cup G_2 V_2 G_2^T V_2) \\ &= (G_1 \cup G_2) (G_1^S \cup G_2^S) \\ &= (G_1 \cup G_2)^2 \end{aligned} \quad = G_B^2$$

So G_B must be a square root of $A_B A_B^S$. Since $A_1 \cup A_2$ is non-singular bimatrix and has non-singular square roots. Among the square roots of $A_B A_B^S$. Then G_B must be secondary symmetric bimatrix. Since $A_B A_B^S$ is secondary symmetric bimatrix and any bipolynomial in a secondary symmetric bimatrix is secondary symmetric bimatrix. Now define $Q_B \equiv G_B^{-1} A_B$ and compute

$$\begin{aligned} Q_B Q_B^S &= (Q_1 \cup Q_2) (Q_1^S \cup Q_2^S) = [(G_1^{-1} \cup G_2^{-1}) (A_1 \cup A_2)] [(G_1^{-1} \cup G_2^{-1}) (A_1 \cup A_2)]^S \\ &= (G_1^{-1} \cup G_2^{-1}) (A_1 \cup A_2) (A_1^S \cup A_2^S) (G_1^S \cup G_2^S)^{-1} \\ &= (G_1^{-1} \cup G_2^{-1}) (G_1 \cup G_2)^2 (G_1^{-1} \cup G_2^{-1}) \\ &= I_1 \cup I_2 = I_B. \end{aligned}$$

- (b) If G_B commutes with Q_B then G_B commutes with Q_B^{-1} , which is a bipolynomial in Q_B . Since $Q_B^{-1} = V_B Q_B^T V_B = Q_B^S$, we have

$$\begin{aligned} A_B^S A_B &= (A_1^S \cup A_2^S) (A_1 \cup A_2) \\ &= (Q_1^S \cup Q_2^S) (G_1 \cup G_2)^2 (Q_1 \cup Q_2) \\ &= (G_1 \cup G_2)^2 (Q_1^S \cup Q_2^S) (Q_1 \cup Q_2) \\ &= (G_1 \cup G_2)^2 \\ &= (A_1 \cup A_2) (A_1^S \cup A_2^S). \end{aligned}$$

Conversely, if $A_B A_B^S = A_B^S A_B$, then

$$\begin{aligned} G_B^2 &= (G_1 \cup G_2)^2 \\ &= (A_1 \cup A_2)(A_1^S \cup A_2^S) \\ &= (A_1^S \cup A_2^S)(A_1 \cup A_2) \\ &= [(G_1 \cup G_2)(Q_1 \cup Q_2)]^S [(G_1 \cup G_2)(Q_1 \cup Q_2)] \\ &= (Q_1^S \cup Q_2^S)(G_1 \cup G_2)^2 (Q_1 \cup Q_2) \\ &= (Q_1^S \cup Q_2^S)(A_1 \cup A_2)(A_1^S \cup A_2^S)(Q_1 \cup Q_2) \\ &= Q_B^S A_B A_B^S Q_B, \end{aligned}$$

So $Q_B A_B A_B^S = A_B A_B^S Q_B$.

That is Q_B commute with $A_B A_B^S$. If G_B is a bipolynomial in $A_B A_B^S$, Q_B must also commutes with G_B .

Theorem 3.17

Let $A_B = A_1 \cup A_2 \in C_{n \times n}$. There exists a complex secondary orthogonal bimatrix $Q_B = (Q_1 \cup Q_2) \in C_{n \times n}$ and complex secondary symmetric bimatrix $G_B = (G_1 \cup G_2) \in C_{n \times n}$ such that $A_B = G_B Q_B$ if and only if $\text{rank}(A_B A_B^S)^k = \text{rank}(A_B^S A_B)^k$, for $k = 1, 2, \dots, n$.

Definition 3.18

A matrix $A_B = A_1 \cup A_2 \in C_{n \times n}$ is called secondary skew orthogonal bimatrix, if $A_B A_B^S = A_B^S A_B = -I_B$. Equivalently $A_B^S = -A_B^{-1}$ or $A_B^{-1} = -A_B^S$.

$$(i.e., (A_1^S \cup A_2^S) = -(A_1^{-1} \cup A_2^{-1}) \text{ or } (A_1^{-1} \cup A_2^{-1}) = -(A_1^S \cup A_2^S)).$$

Theorem 3.19

- (1) If $A_B = (A_1 \cup A_2)$ is secondary skew orthogonal bimatrix of odd order then $\det(A_B) = \pm i$.
- (2) If $A_B = (A_1 \cup A_2)$ is secondary skew orthogonal bimatrix of even order then $\det(A_B) = \pm 1$.
- (3) If $A_B = (A_1 \cup A_2)$ is secondary skew orthogonal bimatrix then $A_B^T, -A_B$ are secondary skew orthogonal bimatrix.

Theorem 3.20

- (1) If $A_B = (A_1 \cup A_2), B_B = (B_1 \cup B_2) \in C_{n \times n}$ are secondary skew orthogonal bimatrix then $A_B B_B$ and $B_B A_B$ are secondary skew orthogonal bimatrix.
- (2) If $A_B = (A_1 \cup A_2)$ is secondary orthogonal (secondary skew orthogonal) bimatrix then iA_B is secondary skew orthogonal (secondary orthogonal) bimatrix.

IV. CONCLUSION

Here, we have studied secondary orthogonal bimatrices as a generalization of orthogonal and secondary orthogonal matrices. Some of the properties and important results of orthogonal and secondary orthogonal matrices are extended to secondary orthogonal bimatrices.



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